THE CAMBRIDGE MATHEMATICAL JOURNAL.

VOL. III.] MAY, 1843. [No. XVIII.

I.—ON CERTAIN FORMULÆ MADE USE OF IN PHYSICAL ASTRONOMY.

SOME of the most important results of Physical Astronomy may be arrived at very readily by means of the following propositions, which are here demonstrated a little differently from the common method. In a future paper it is intended to apply the equations here obtained to the case of the Moon disturbed by the Sun, and of a planet disturbed by another planet.

1. *To obtain the polar formulæ for the forces which act upon a particle* (m) *moving in one plane.*

Let r be the radius vector of m at the time t, ω the angular velocity of r, and ϕ the angle which r makes with any fixed line (L) drawn in the plane of motion through the origin. Then, $\frac{dr}{dt}$ and $r\omega$ being the velocities of m along and perpendicular to r, the velocity of m parallel to (L) will be

$$\frac{dr}{dt}\cos\phi + r\omega\sin\phi\,;$$

(we suppose (L) to lie farther from the prime radius than r,) therefore the force on m parallel to (L) will be

$$\frac{d}{dt}\left(\frac{dr}{dt}\cos\phi + r\omega\sin\phi\right)$$

$$= \left(\frac{d^2r}{dt^2} - r\omega^2\right)\cos\phi + \left\{\frac{dr}{dt}\omega + \frac{d(r\omega)}{dt}\right\}\sin\phi \ldots\ldots (1);$$

observing that $\frac{d\phi}{dt} = -\omega$, since ($L$) is fixed and farther from the prime radius than r.

Now the position of (L) is arbitrary, therefore we may choose it so that at any time (t) it shall coincide with, or be perpendicular to r. Let (L) therefore coincide with r, i.e. let $\phi = 0$; then (1), which is now the force along r, becomes

$$\frac{d^2r}{dt^2} - r\omega^2.$$

Again, let (L) be perpendicular to r, i.e. let $\phi = \frac{\pi}{2}$; then (1), which now is the force perpendicular to r, becomes

$$\frac{dr}{dt}\,\omega + \frac{d\,(r\omega)}{dt} \qquad \text{or } \frac{1}{r}\,\frac{d\,(r^2\omega)}{dt}.$$

Hence, if at any time (t), P be the force on m along r, and Q that perpendicular to r, we have

$$\text{(A)} \begin{cases} \dfrac{d^2r}{dt^2} - r\omega^2 = P \ldots\ldots\ldots (2). \\ \dfrac{1}{r}\,\dfrac{d\,(r^2\omega)}{dt} = Q \ldots\ldots\ldots (3). \end{cases}$$

2. *If* T *be the force on* m *along its path, and* N *that perpendicular to it, to find* T *and* N.

Let v be the velocity of m, and ϕ the angle which (L) makes with the direction of v; then the velocity of m parallel to (L) is $v\cos\phi$, and therefore the force on m along (L) is

$$\frac{d\,(v\cos\phi)}{dt},$$

$$\text{or} \quad \frac{dv}{dt}\cos\phi - v\sin\phi\,\frac{d\phi}{dt} \ldots\ldots\ldots (4).$$

Now if ψ be the angle which the tangent to the path of m makes with the prime radius, we have $d\phi = -d\psi$ (we suppose (L) to make a greater angle with the prime radius than the tangent does), and $d\psi = \frac{ds}{\rho}$, ρ being the radius of curvature, and s the arc of the path of m:

$$\text{hence } \frac{d\phi}{dt} = -\frac{1}{\rho}\,\frac{ds}{dt} = -\frac{v}{\rho}, \text{ and therefore (4) becomes}$$

$$\frac{dv}{dt}\cos\phi + \frac{v^2}{\rho}\sin\phi.$$

Hence putting $\phi = 0$ and $\frac{\pi}{2}$, as before, we find

$$\frac{dv}{dt} = T \qquad \frac{v^2}{\rho} = N.$$

3. The equations (A) in art. (1) may be transformed in the following manner.

Assume $h = r^2 \frac{d\theta}{dt}$ and $\frac{1}{u} = r$,

then $$\frac{dr}{dt} = \frac{d}{d\theta}\left(\frac{1}{u}\right)\frac{d\theta}{dt} = -\frac{du}{d\theta}\frac{1}{u^2}\frac{d\theta}{dt}$$

$$= -\frac{du}{d\theta}h;$$

$$\therefore \frac{d^2r}{dt^2} = -\frac{d^2u}{d\theta^2}\frac{d\theta}{dt}h - \frac{du}{d\theta}\frac{dh}{dt}$$

$$= -\frac{d^2u}{d\theta^2}h^2u^2 - \frac{du}{d\theta}\frac{Q}{u};$$

since $\frac{dh}{dt} = \frac{Q}{u}$ by (3).

Hence (2) becomes

$$P = -\frac{d^2u}{d\theta^2}h^2u^2 - \frac{du}{d\theta}\frac{Q}{u} - h^2u^3;$$

or $$\frac{d^2u}{d\theta^2} + u + \frac{du}{d\theta}\frac{Q}{h^2u^2} + \frac{P}{h^2u^2} = 0 \ldots\ldots (5).$$

Also, since $$\frac{dh}{dt} = \frac{dh}{d\theta}\frac{d\theta}{dt} = \frac{dh}{d\theta}hu^2,$$

(3) becomes $$h\frac{dh}{d\theta} = \frac{Q}{u^3};$$

and therefore $$h^2 = 2\int\frac{Q}{u^3}d\theta \ldots\ldots\ldots\ldots (6).$$

(5) and (6) are the two equations made use of in the Lunar theory: the value of h^2 given in (6) is usually substituted in (5).

4. For a body acted on by a central force varying inversely as the square of the distance, we have

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{\mu}{r^2} \ldots\ldots\ldots\ldots (1),$$

and hence h = constant. $$\frac{dh}{dt} = 0;$$

Also, putting $r = \frac{1}{u}$, we have

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta},$$

$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2};$$

therefore (1) becomes

$$-h^2u^2\frac{d^2u}{d\theta^2}-h^2u^3=-\mu u^2,$$

$$\text{or}\quad \frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2},$$

$$\text{whence}\quad u=\frac{\mu}{h^2}+A\cos(\theta+B),$$

which shews that the orbit is a conic section, the pole being the focus.

5. To determine this conic section completely when the circumstances of projection are given, we have from (1), multiplying by $\frac{dr}{dt}$, and integrating,

$$\left(\frac{dr}{dt}\right)^2+\frac{h^2}{r^2}=\frac{2\mu}{r}+C\ldots\ldots\ldots(2).$$

Now $$\left(\frac{dr}{dt}\right)^2+\frac{h^2}{r^2}=\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2=\left(\frac{ds}{dt}\right)^2=v^2;$$

$$\text{therefore}\quad v^2=\frac{2\mu}{r}+C\ldots\ldots\ldots\ldots\ldots(3).$$

When the particle is at either apse, $\frac{dr}{dt}=0$, and therefore by (2)

$$\frac{h^2}{r^2}=\frac{2\mu}{r}+C;$$

$$\text{or}\quad r^2+\frac{2\mu}{C}r-\frac{h^2}{C}.$$

Now the roots of this equation are $a(1-e)$ and $a(1+e)$, therefore $-\frac{2\mu}{C}$ = their sum $=2a$, and therefore $C=-\frac{\mu}{e}$, and $-\frac{h^2}{C}$ = their product $=a^2(1-e^2)$, and $\therefore h^2=a\mu(1-e^2)$.

Hence we have

$$v^2=\frac{2\mu}{r}-\frac{\mu}{a}\ldots\ldots\ldots\ldots\ldots(\alpha),$$

$$h^2=a\mu(1-e^2)\ldots\ldots\ldots\ldots(\beta),$$

$$\text{and}\quad r=\frac{a(1-e^2)}{1+e\cos(\theta-\pi)}\ldots\ldots(\gamma).$$

Let u, c, and β, be the velocity, distance, and angle of projection; and take c to be the prime radius; then, when

$\theta = 0$, $r = c$, $v = u$: also we have $h = uc \sin \beta$. Hence, (α), (β), (γ), give

$$\frac{1}{a} = \frac{2\mu}{c} - u^2,$$

$$e^2 = 1 - \frac{u^2c^2 \sin^2 \beta}{a\mu},$$

$$\cos \pi = \frac{a(1 - e^2) - c}{ec},$$

which equations give a, e, and π.

6. A particle m moves anyhow in space: if P be the force on it along its radius vector r, Q that perpendicular to r in the plane of the orbit of m (*i.e.* the plane containing r and the direction of the motion of m at any time), and S the force perpendicular to this plane; *it is required to determine expressions for* P, Q, *and* S.

Let ω, as before, be the angular velocity of r, and η the angular velocity of the orbit round r. Let any fixed line (L) make an angle ϕ with r, and let the plane containing L and r make an angle ψ with the plane of the orbit. Then, the velocities of m being $\frac{dr}{dt}$ and $r\omega$, and the resolved part of $r\omega$ in the plane (Lr) being $r\omega \cos \psi$, the velocity parallel to (L) will be

$$\frac{dr}{dt} \cos \phi + r\omega \cos \psi \sin \phi,$$

and therefore the force on m, parallel to L, will be

$$\left(\frac{d^2r}{dt^2} + r\omega \frac{d\phi}{dt} \cos \psi\right) \cos \phi$$

$$+ \left(-\frac{dr}{dt}\frac{d\phi}{dt} + \frac{d(r\omega)}{dt} \cos \psi\right) \sin \phi - r\omega \sin \psi \sin \phi \frac{d\psi}{dt};$$

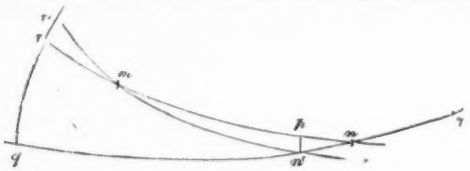
and this expression, if we put $\psi = 0$, $\phi = 0$, becomes P; if $\psi = 0$, $\phi = \frac{\pi}{2}$, it becomes Q; and if $\psi = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$, it becomes S. Moreover if $\psi = \frac{\pi}{2}$, $\frac{d\phi}{dt} = 0$, and $\frac{d\psi}{dt} = -\eta$, and if $\psi = 0$, $\frac{d\phi}{dt} = -\omega$. Hence we have

$$\frac{d^2r}{dt^2} - r\omega^2 = P,$$

$$\frac{1}{r}\frac{d(r^2\omega)}{dt} = Q,$$

$$r\omega\eta = S.$$

7. Let γnq (see figure) be the plane of the ecliptic, $n\,m$ the orbit of m; let $\theta = \gamma n + nm$, i.e. the longitude of m on its



orbit, and $\nu = \gamma n$ the longitude of the node. Then ωdt, the small angle described by r in the time dt, will be equal to the increment of mn + the motion of n in the time dt resolved along mn; i.e.

$$\omega dt = d(\theta - \nu) + d\nu \cos \iota,$$

ι being the inclination of the orbit to the ecliptic, therefore

$$\omega = \frac{d\theta}{dt} - 2 \sin^2 \frac{\iota}{2} \frac{d\nu}{dt}.$$

8. Let mn' be a consecutive position of the plane of the orbit, draw $n'p$ perpendicular to mn, and qrr' perpendicular to γnq, qn' being $\frac{1}{2}\pi$; then the angle $pmn' = \eta dt$, $n'n = d\nu$, and $rr' = d\iota$; therefore

$$d\nu = \frac{pn'}{\sin \iota} = \frac{\eta dt \sin (\theta - \nu)}{\sin \iota},$$

$$d\iota = \eta dt \sin mr = \eta dt \cos (\theta - \nu);$$

$$\therefore \quad \frac{d\nu}{dt} = \frac{\sin (\theta - \nu)}{\sin \iota} . \eta, \quad \frac{d\iota}{dt} = \cos (\theta - \nu) . \eta.$$

9. Hence, if we put $h = r^2\omega = r^2 \left(\frac{d\theta}{dt} - 2 \sin^2 \frac{\iota}{2} \frac{d\nu}{dt} \right)$ we have the following general equations to determine the quantities r, θ, ν, ι,

$$\left.\begin{aligned} \frac{d^2r}{dt^2} - \frac{h^2}{r^3} &= P, \\ \frac{dh}{dt} &= Qr, \\ \frac{d\nu}{dt} &= \frac{Sr}{h} \frac{\sin (\theta - \nu)}{\sin \iota}, \\ \frac{d\iota}{dt} &= \frac{Sr}{h} \cos (\theta - \nu), \\ \text{where} \quad h &= r^2 \left(\frac{d\theta}{dt} - 2 \sin^2 \frac{\iota}{2} \frac{d\nu}{dt} \right), \end{aligned}\right\}$$

r is the radius vector of m, θ is the longitude of m on its

orbit, ν the longitude of the node, and ι the inclination of the orbit.

10. Let M, m, m' be a free system of three mutually attracting bodies: then we may suppose one of them (M) to become fixed, provided we apply forces equal and opposite to those which act on M to the whole system. Let r be the distance Mm, $r' = Mm'$, $r_1 = mm'$, then the forces which act on m will be

$$\frac{M}{r^2} \text{ along } mM, \qquad \frac{m'}{r_1^2} \text{ along } mm',$$

$$-\frac{m}{r^2} \text{ along } mM, \qquad -\frac{m'}{r'^2} \text{ parallel to } Mm'.$$

("Along Mm" means from M towards m, "along mM" means from m towards M.)

Now let Pm $(= s)$ be any curve (not necessarily in the plane Mmm'); then, if F be the force which acts on m along ds, we have

$$F = -\frac{M+m}{r^2}\cos(r, ds) - \frac{m'}{r_1^2}\cos(r_1, ds) - \frac{m'}{r'^2}\cos(r', ds).$$

$$\text{Now} \quad \cos(r, ds) = \frac{dr}{ds}, \qquad \cos(r_1, ds) = \frac{dr_1}{ds},$$

$$\text{and} \quad \cos(r', ds) = \frac{\text{projection of } ds \text{ on } Mm'}{ds};$$

$$= \frac{d\{r\cos(r, r')\}}{ds}.$$

$$\text{Hence} \quad F = -\frac{M+m}{r^2}\frac{dr}{ds} - \frac{m'}{r_1^2}\frac{dr_1}{ds} - \frac{m'}{r'^2}\frac{d\{r\cos(r, r')\}}{ds},$$

$$= \frac{d}{ds}\left(\frac{M+m}{r} + \frac{m'}{r_1} - \frac{m'r\cos(r, r')}{r'^2}\right).$$

And if we assume

$$M + m = \mu, \text{ and } R = m'\left\{\frac{1}{r_1} - \frac{r}{r'^2}\cos(r, r')\right\},$$

we have

$$F = \frac{d}{ds}\left(\frac{\mu}{r} + R\right);$$

In this differentiation we are to suppose m' fixed, and m to be displaced along s.

11. Hence we may find the forces P, Q, S, which act on m, (see Art. 6) as follows.

1st. Let s coincide with r, then $ds = dr$ and $F = P$; therefore

$$P = -\frac{\mu}{r^2} + \frac{dR}{dr}.$$

2ndly. Let s meet r at right angles and lie in the plane of the orbit of m; then $ds = rd\theta$, r does not vary, and $F = Q$; therefore

$$Q = \frac{dR}{rd\theta}.$$

3rdly. Let s meet r at right angles, and be perpendicular to the plane of m's orbit; then $ds = r \sin(\theta - \nu)\, d\iota$, r does not vary, and $F = S$; therefore

$$S = \frac{1}{r \sin(\theta - \nu)} \frac{dR}{d\iota}.$$

It is important to remember, that in these formulæ, $\frac{dR}{dr}$ is the differential coefficient of R on the supposition that θ, ν, ι, are constant, $\frac{dR}{d\theta}$ on the supposition that r, ν, ι, are constant, and $\frac{dR}{d\iota}$ on the supposition that r, θ, ν, are constant.

12. Hence we have the following equations to determine the motion of M:

$$\left.\begin{aligned} \frac{d^2r}{dt^2} - \frac{h^2}{r^3} &= -\frac{\mu}{r^2} + \frac{dR}{dr}, \\ \frac{dh}{dt} &= \frac{dR}{d\theta}, \\ \frac{d\nu}{dt} &= \frac{1}{h \sin \iota} \frac{dR}{d\iota}, \\ \frac{d\iota}{dt} &= \frac{1}{h} \cot(\theta - \nu) \frac{dR}{d\iota}, \\ \text{where } h &= r^2 \left(\frac{d\theta}{dt} - 2 \sin^2 \frac{\iota}{2} \frac{d\nu}{dt} \right). \end{aligned}\right\}$$

13. If we suppose s to coincide with the actual path of m, $ds = vdt$, and by Art. (2) we have

$$F(= T) = \frac{1}{v} \left(-\frac{\mu}{r^2} \frac{dr}{dt} + \frac{dR}{dt} \right),$$

and hence

$$v \frac{dv}{dt} = -\frac{\mu}{r^2} \frac{dr}{dt} + \frac{dR}{dt};$$

therefore

$$v^2 = \frac{2\mu}{r} + \int \frac{dR}{dt}\, dt,$$

$\frac{dR}{dt}$ is the differential of R with respect to t, supposing that r and θ alone are functions of t.

14. To find R, let λ and λ' be the latitudes of m and m', and θ_1 θ'_1 the longitudes of m and m' measured on the ecliptic; then

$$\cos(r, r') = \sin\lambda \sin\lambda' + \cos\lambda \cos\lambda' \cos(\theta_1' - \theta_1),$$

$$r_1^2 = r^2 + r'^2 - 2rr' \cos(r, r'),$$

$$R = m' \left\{ \frac{1}{r_1} - \frac{r}{r'^2} \cos(r, r') \right\}$$

$$\tan(\theta_1 - \nu) = \cos\iota \tan(\theta - \nu) \quad \tan(\theta_1' - \nu') = \cos\iota' \tan(\theta' - \nu')$$

$$\sin\lambda = \sin\iota \sin(\theta - \nu) \quad \sin\lambda' = \sin\iota' \sin(\theta' - \nu').$$

M. O. B.

II.—NOTE ON VIBRATING CORDS.

THE equilibrium of a stretched cord is disturbed between the limits $x = l$ and $x = -l$, in such a manner as to cause a pulse, whose length is $2l$, to run along the cord in the positive direction: to find the portion of the resulting disturbance at any future time t, which is due to the disturbance of each element of the cord between $x = -l$ and $x = l$.

Let x and x' be the distances of two points, P and P', in the cord from the origin, the latter being between the limits $-l$ and l. Then $A \cos p(x - x' - at)$ satisfies the differential equation of the motion of the cord, and represents a series of waves, which we may suppose to originate from the point P', proceeding in the positive direction; and therefore, taking the sum of an infinite number of such systems, we find

$$\int_{-\infty}^{\infty} A \cos p(x - x' - at)\, dp,$$

for a disturbance proceeding in the positive direction from P'. But, if $A = 1$, the value of this integral is 0, except when

$$x - x' = at;$$

and it therefore represents an infinitely short pulse, leaving P', when $t = 0$. The total disturbance resulting from such pulse leaving every point P', between $x' = -l$ and $x' = l$, is

$$\int_{-l}^{l} Fx'\, dx' \int_{-\infty}^{\infty} \cos p(x - x' - at)\, dp;$$

and that this may represent the result of the given disturbance, it must be equal to the given initial disturbance, which we shall call fx', when $t = 0$. Hence, to determine Fx', we have

$$\int_{-l}^{l} Fx'\, dx' \int_{-\infty}^{\infty} \cos p(x - x')\, dp = fx'.$$

Hence, by Fourier's theorem, $Fx' = \frac{1}{\pi} fx'$, when x' is between $-l$ and l, and $Fx' = 0$, when x' is not between these limits. The part of the resulting disturbance, due to the point P', is therefore

$$\frac{1}{\pi} fx'dx' \int_{-\infty}^{\infty} \cos p\,(x - x' - at)\,dp.$$

The known result of a pulse in a vibrating cord, as it exists at any given period of time, has been thus expressed as the sum of a number of disturbances due to each part of the original pulse. The same method might possibly be applied to find the unknown result of an element of the surface of a pulse propagated in an elastic medium, such as air, or the ether which produces light, the disturbance of the element being supposed to diverge outwards in every direction, so as to form a spherical wave.

III.—ON THE INTEGRATION OF CERTAIN EQUATIONS OF FINITE DIFFERENCES.

By B. Bronwin.

The methods which I have employed in the integration of certain differential equations in two papers printed in this *Journal*, may be applied also to similar equations of finite differences. Without giving any general theory, I shall illustrate the subject in the present paper by a few examples.

Let $(a + bx)\,\Delta^2 y_x + (h + kx)\,\Delta y_x - my_x = 0,$

where $m = rk$, r being an integer. Taking the difference making $a_1 = a + b$, $h_1 = h + k + b$, $m_1 = m - k$, we have

$$(a_1 + bx)\,\Delta^2 y_x + (h_1 + kx)\,\Delta^2 y_x - m_1 \Delta y_x = 0.$$

Repeating this process r times, we at length find, since $m_r = 0$,

$$(a_r + bx)\,\Delta^{r+2} y_x + (h_r + kx)\,\Delta^{r+1} y_x = 0.$$

If $a_r = l$, $a_r - h_r = f$, $b - k = g$; the last equation may be put under the form

$$(l + bx)\,\Delta^{r+1} y_{x+1} - (f + gx)\,\Delta^{r+1} y_x = 0.$$

Making $\frac{f + gx}{l + bx} = \omega_x$, and integrating there results

$$\Delta^{r+1} y_x = CP\omega_{x-1}, \quad \Delta^r y_x = C\Sigma P\omega_{x-1} + C'.$$

Now, from the r equations obtained by taking successively the difference, we may eliminate Δy_x, $\Delta^2 y_x$, &c.; and shall have

$$y_x = M\Delta^r y_x + N\Delta^{r+1} y_x = C(M\Sigma P\omega_{x-1} + NP\omega_{x-1}) + C'M,$$

where C and C' are arbitraries, and M and N will be known

functions of x. This is the complete integral. M and N may sometimes be found by series precisely as in my first paper on differential equations.

Suppose $\Delta^2 y_x + kx\Delta y_x - 2ky_x = 0$; then $\omega_x = 1 - 2k - kx$, and $y_x = C\{(1 + kx + kx^2)\,\Sigma P\omega_{x-1} + xP\omega_{x-1}\} + C'(1 + kx + kx^2)$.

Let $x\Delta^2 y_x + kx\Delta y_x - 2ky_x = 0$. Here $\omega_x = \dfrac{2k + (k-1)x}{2 + x}$,

and $y_x = C\{(2 + k + kx)\,x\Sigma(-1)^x P\omega_{x-1} + (1 + x)\,x\,(-1)^x P\omega_{x-1}\} + C'(2 + k + kx)\,x$.

Let $(a + bx + cx^2)\,\Delta^2 y_x + (h + kx)\,\Delta y_x - my_x = 0$.

Taking the difference, making

$a_1 = a + b + c$, $b_1 = b + 2c$, $h_1 = h + k + b + c$, $k_1 = k + 2c$, $m_1 = m - k$;

we have $(a_1 + b_1 x + cx^2)\,\Delta^3 y_x + (h_1 + k_1 x)\,\Delta^2 y_x - m_1 \Delta y_x = 0$.

If $m = rk + r(r-1)c$, repeating the process r times, we shall ultimately have

$$(a_r + b_r x + cx^2)\,\Delta^{r+2} y_x + (h_r + k_r x)\,\Delta^{r+1} y_x = 0.$$

Make $$\frac{a_r - h_r + (b_r - k_r)\,x + cx^2}{a_r + b_r x + cx^2} = \omega_x,$$

and the last equation may be put under the form

$$\Delta^{r+1} y_{x+1} - \omega_x \Delta^{r+1} y_x = 0,$$

which integrated gives

$$\Delta^{r+1} y_x = CP\omega_{x-1}, \quad \Delta^r y_x = C\Sigma P\omega_{x-1} + C'.$$

Eliminating Δy_x, $\Delta^2 y_x$, &c. as before, we find

$$y_x = M\Delta^r y_x + N\Delta^{r+1} y_x = C(M\Sigma P\omega_{x-1} + NP\omega_{x-1}) + C'M,$$

where M and N denote known functions of x, which result from the elimination.

As a more particular example, let

$$x(1 + x)\,\Delta^2 y_x + k(1 - x)\,\Delta y_x + ky_x = 0;$$

then $(2 + 3x + x^2)\,\Delta^3 y_x + \{2 - (k-2)\,x\}\,\Delta^2 y_x = 0$;

or $(2 + 3x + x^2)\,\Delta^2 y_{x+1} - \{(k+1)\,x + x^2\}\,\Delta^2 y_x = 0$,

$$\frac{(k+1)\,x + x^2}{2 + 3x + x^2} = \omega_x,$$

$$\Delta^2 y_x = CP\omega_{x-1}, \quad \Delta y_x = C\Sigma P\omega_{x-1} + C',$$

and $y_x = C(1 - x) + C'\left\{(1 - x)\,\Sigma P\omega_{x-1} + \dfrac{1}{k}(x + x^2)\,P\omega_{x-1}\right\}$.

In the preceding examples the coefficient of the last term is constant, but it is not necessary that it should be so. Thus the equation $A_x\Delta^2 y_x + (a - x)\,B_x\Delta y_x + B_x y_x = 0$

is integrable after taking the difference once, the last term of the result vanishing when y_x is eliminated.

We shall now give a few examples where successive integration must be employed.

Let $(a + bx)\,\Delta^2 y_x + (h + kx)\,\Delta y_x + my_x = 0,\ m = rk.$

Integrating each term, and making

$$a_1 = a - b,\ h_1 = h - b - k,\ m_1 = m - k,$$

we have $(a_1 + bx)\,\Delta y_x + (h_1 + kx)\,y_x + m_1 \Sigma y_x + C = 0.$

Or if $\Sigma y_x + \dfrac{C}{m_1} = y'_x$, by substitution the last equation becomes

$$(a_1 + bx)\,\Delta^2 y'_x + (h_1 + kx)\,\Delta y'_x + m_1 y_x = 0.$$

Repeating this process r times, we shall at last find

$$(a_r + bx)\,\Delta^{r-1} y_x + (h_r + kx)\,y_x^{r-1} + C = 0,$$

which is equivalent to

$$y_{x+1}^{r-1} - \omega_x y_x^{r-1} + \frac{C}{a_r + bx} = 0,\quad \omega_x = \frac{a_r - h_r + (b - k)\,x}{a_r + bx}.$$

Thus y_x^{r-1} is known; from which we easily find in succession y_x^{r-2}, y_x^{r-3}, &c. and y_x, merely by the operation of differencing. And as y_x will contain two arbitraries, it will be the complete integral.

As a more particular example, let

$$x\Delta^2 y_x + kx\Delta y_x + 2ky_x = 0,\ \text{and make } y'_x = \Sigma y_x + \frac{C}{k};$$

then we find successively

$$(x - 1)\,\Delta y_x + \{k\,(x - 1) - 1\}\,y_x + k\Sigma y_x + C = 0;$$
$$(x - 1)\,\Delta^2 y'_x + \{k\,(x - 1) - 1\}\,\Delta y'_x + ky'_x = 0;$$
$$(x - 2)\,\Delta y'_x + \{k\,(x - 2) - 2\}\,y'_x = C,$$

or if $\omega_x = \dfrac{x}{x - 2} - k,\ y'_{x+1} - \omega_x y'_x = \dfrac{C}{x - 2};$

therefore $y'_x = CP\omega_{x-1}\Sigma \dfrac{1}{(x - 2)\,P\omega_x} + C'P\omega_{x-1},$

and consequently

$$y_x = C\Delta\left\{P\omega_{x-1}\Sigma \frac{1}{(x - 2)\,P\omega_x}\right\} + C'\Delta P\omega_{x-1}.$$

Next let $(a + bx + cx^2)\,\Delta^2 y_x + (h + kx)\,\Delta y_x + my_x = 0.$

Integrating every term, there results

$$(a_1 + b_1 x + cx^2)\,\Delta y_x + (h_1 + k_1 x)\,y_x + m_1 \Sigma y_x + C = 0;$$
$$a_1 = a - b + c,\ b_1 = b - 2c,\ h_1 = h - b + 3c - k,\ k_1 = k - 2c,$$
$$m_1 = m - k + 2c.$$

Make $$\Sigma y_x + \frac{C}{m_1} = y'_x;$$

then the last, by substitution, becomes

$$(a_1 + b_1 x + cx^2)\,\Delta^2 y'_x + (h_1 + k_1 x)\,\Delta y'_x + m_1 y'_x = 0.$$

If $m = rk - r(r+1)c$, we shall have, by repeating the integration r times,

$$(a_r + b_r x + cx^2)\,\Delta y_x^{r-1} + (h_r + k_r x)\,y_x^{r-1} + C = 0,$$

which is integrable, and hence the integral of the proposed is readily found.

The method of series will apply to equations of finite differences similarly as to differential equations, factorials supplying the place of powers.

If $$y_x = A_0 + A_1 x + A_2 x x_{-1} + A_3 x x_{-1} x_{-2} + ;$$

$\Delta y_x = A_1 + 2A_2 x + 3A_3 x x_{-1} + ;$ and $\Delta^2 y_x = 2A_2 + 2.3A_3 x + ;$

where $$x_{-1} = x - 1,\ x_{-2} = x - 2,\ \&c.$$

Now let $$(a + bx)\,\Delta^2 y_x - bx\Delta y_x + hy_x = 0.$$

The scale of this is

$$(n+2)(n+1)(a+nb)\,A_{n+2} + (h - nb)\,A_n = 0;$$

from which we readily find

$$y_x = A_0\left\{1 - \frac{h}{2a} x x_{-1} + \frac{h(h-2b)}{2.3.4a(a+2b)} x x_{-1} x_{-2} x_{-3} - \frac{h(h-2b)(h-4b)}{2.3.4.5.6a(a+2b)(a+4b)} x \ldots . x_{-5} + \right\}$$

$$+ A_1\left\{x - \frac{h-b}{2.3(a+b)}\, x x_{-1} x_{-2} + \frac{(h-b)(h-3b)}{2.3.4.5(a+b)(a+3b)}\, x x_{-1} x_{-2} x_{-3} x_{-4} - \right\},$$

the complete integral, having two arbitraries A_0 and A_1. But one of these series will fail when $a + ib = 0$, i an integer. In this case we find, making

$$n = 2 - \frac{a}{b},\ 4 - \frac{a}{b},\ \&c.;\ A_{4-\frac{a}{b}} = -\frac{b(h - 2b + a)}{2(a - 3b)(a - 4b)}\, A_{2-\frac{a}{b}},$$

$$A_{6-\frac{a}{b}} = \frac{b^2(h - 2b + a)(h - 4b + a)}{2.4(a - 3b)(a - 4b)(a - 5b)(a - 6b)}\, A_{2-\frac{a}{b}},\ \&c.;$$

which will give a new series to supply the defective one, the first term of which is

$$A_{2-\frac{a}{b}} \cdot x x_{-1} \cdot \ldots \ldots \ldots x_{-1+\frac{a}{b}}.$$

We find for a descending series, making $a+h=k$ to abridge,

$$y_x = A_{\frac{k}{b}}\, x x_{-1} \ldots x_{-\frac{h}{b}+1}\left\{1 - \frac{h(h-b)(k-2b)}{2b^2(bx - h + b)}\right.$$

$$\left. + \frac{h(h-b)(h-2b)(h-3b)(k-2b)(k-4b)}{2 \cdot 4b^4(bx - h + b)(bx - h + 2b)} - \right\}.$$

But this is only a particular integral, and we cannot find the other particular integral by a descending series.

Let $x(x+1)\Delta^2 y_x - kx\Delta y_x + my_x = 0,$

where $k = p + q - 1, \quad m = pq.$ Make $y_x = \Sigma A_n x x_{-1} \ldots x_{-n+1}.$

Then $\Delta y_x = \Sigma n A_n x x_{-1} \ldots x_{-n+2}, \quad \Delta^2 y_x = \Sigma n n_{-1} A_n x x_{-1} \ldots x_{-n+3}.$

We find the scale to be

$$n_2 n_1^2 n A_{n+2} + n_1 n (2n - k) A_{n+1} + (p - n)(q - n) A_n = 0.$$

Here $n_1 = n + 1, \quad n_2 = n + 2, \quad n_{-1} = n - 1,$ &c. to abridge.

Making $n = p$, $p - 1$, $p - 2$, &c., $A_{p+1} = 0$; and then $n = q$, $q - 1$, &c., we obtain

$$y_x = A_p x x_{-1} \ldots x_{-p+1} \left\{1 + \frac{p(p-1)}{x-p+1} + \frac{p(p-1)^2(p-2)}{2(x-p+1)(x-p+2)} + \text{&c.}\right\}$$

$$+ A_q x x_{-1} \ldots x_{-q+1} \left\{1 + \frac{q(q-1)}{x-q+1} + \frac{q(q-1)^2(q-2)}{2(x-q+1)(x-q+2)} + \text{&c.}\right\}$$

When p and q are not integers, the factorials $xx_{-1} \ldots x_{-p+1}$, $xx_{-1} \ldots x_{-q+1}$ can only be approximately calculated.

The above could be integrated by successively taking the difference when p or q is an integer, as the last term would vanish after p or q operations.

Making $n = 0$, 1, 2, &c., we might obtain two ascending series. But they would not terminate when p or q is integer. Possibly however, by changing the arbitraries for others, series that terminate in the case supposed might be obtained.

If $(a + bx)\Delta^2 y_x + (h + kx)\Delta y_x - rky_x = 0$, r a positive integer,

and $y_x = \Sigma A_n x x_{-1} \ldots x_{-n+1}$;

the scale is

$$n_2 n_1 (a + nb) A_{n+2} + n_1 (h + nb + nk) A_{n+1} + (n - r) k A_n = 0.$$

From this we find

$$A_{r-1} = rB_0 A_r, \quad A_{r-2} = rr_{-1} B_1 A_r, \quad A_{r-3} = rr_{-1} r_{-2} B_2 A_r, \text{ &c.};$$

B_0, B_1, B_2, &c. being put for known quantities. We shall therefore have a particular integral in a descending series which will terminate. This will be the expression for $C'M$ in our first equation.

We may find the complete integral in two ascending series, having A_0 and A_1 for arbitraries; but neither of these series will terminate. This may seem a little paradoxical, but perhaps will admit of explanation. I shall not however attempt to give any, from fear of extending this paper to too great a length.

Let $y_{x+2} + (a + bx)\, y_{x+1} + (h + kx)\, y_x = 0.$

Make $y_x = m^x \sin(\pi x + \beta) z_x$, where π is 3.14159, and β an arbitrary constant. Substituting this quantity, the proposed becomes

$$m^2 z_{x+2} - m(a + bx) z_{x+1} + (h + kx) z_x = 0,$$

or

$$\Delta^2 z_x + (f - gx)\, \Delta z_x + l z_x = 0,$$

making $m = \frac{k}{b}$, $f = 2 - \frac{ab}{k}$, $g = \frac{b^2}{k}$, $l = 1 - \frac{ab}{k} + \frac{b^2 h}{k^2}$.

Here a particular integral suffices on account of the arbitrary β. And if l be a multiple of g positive or negative, we shall have one in finite terms. If we cannot have a series which terminates, we may assign any value to one of the two arbitraries, in order to simplify the resulting series.

Let $(a + bx)\, y_{x+2} + (f + gx)\, y_{x+1} + (h + kx)\, y_x = 0.$

The same substitution, as in the last example, will give

$$(a + bx)\, \Delta^2 z_x + (f_1 + g_1 x)\, \Delta z_x + k_1 z_x = 0.$$

$$2a - \frac{f}{m} = f_1, \quad 2b - \frac{g}{m} = g_1, \quad a - \frac{g}{m} + \frac{h}{m^2} = k_1,$$

and $$bm^2 - gm + k = 0.$$

The two values of m give two equations; a particular integral of either will suffice.

In this last example we might have made $y_x = m^x z_x$; and a particular integral of each of the resulting equations, if distinct, would give the complete integral.

Instead of forming the scale, we might proceed thus: make

$$y_x = A_0 + A_1 x + A_2 x(x - 1) +, \text{ or}$$

$$y_x = A_0 + A_1 \frac{x}{p + qx} + A_2 \frac{x(x - 1)}{(p + qx)(p_1 + q_1 x)} +, \text{ or}$$

$$y_x = A_0 f_1(x) + A_1 x f_2(x) + A_2 x(x - 1) f_3(x) +;$$

$f_1(x)$, $f_2(x)$, denoting any functions of x.

Make $x = 0, 1, 2,$ &c., successively, and we have A_0, A_1, A_2, &c. in terms of y_0, y_1, y_2, &c. But the proposed equation gives y_2, y_3, &c. in terms of y_0, y_1. We have therefore the coefficients in terms of these last quantities.

From these assumed series we may find results commencing at any term, and forming either an ascending or descending series.

[*Note*—In my last paper, p. 182,

$$y = C\int_0^1 t^{q-1} dt (1-t)^{q-1} \cos(2rxt - rx) + C'\int_0^1 t^{q-1} dt (1-t)^{q-1} \sin(2rxt - rx),$$

is erroneously given as the complete integral of

$$x\frac{d^2y}{dx^2} + 2q\frac{dy}{dx} + r^2xy = 0.$$

The second integral vanishes between the limits.]

IV.—ON A CLASS OF DIFFERENTIAL EQUATIONS, AND ON THE LINES OF CURVATURE OF AN ELLIPSOID.

By A. Cayley, B.A. Fellow of Trinity College.

Consider the primitive equation

$$fx + gy + hz + \ldots = 0 \ldots\ldots\ldots\ldots (1),$$

between (n) variables x, y, z, the constants f, g, h being connected by the equation

$$H(f, g, h \ldots) = 0 \ldots\ldots\ldots\ldots (2),$$

H denoting a homogeneous function. Suppose that $f, g, h \ldots$ are determined by the conditions

$$fx_1 + gy_1 + hz_1 \ldots = 0 \ldots\ldots\ldots\ldots (3),$$

$$fx_{n-2} + gy_{n-2} + hz_{n-2} \ldots = 0.$$

Then writing

$$x = \begin{vmatrix} y, z & \ldots \\ y_1, z_1 & \ldots \\ \vdots & \vdots \\ y_{n-2} z_{n-2} & \ldots \end{vmatrix} \ldots\ldots\ldots\ldots\ldots (4).$$

With analogous expressions for $y, z \ldots$ the equations (3) give $f, g, h, \ldots$ proportional to $x, y, z, \ldots$ or eliminating $f, g, h \ldots$ by the equation (2),

$$H(x, y, z \ldots) = 0 \ldots\ldots\ldots\ldots (5).$$

Conversely the equation (5), which contains, in appearance, $n(n-2)$ arbitrary constants, is equivalent to the system (1) (2). And if H be a rational integral function of the order (r), the first side of the equation (5) is the product of r factors, each of them of the form given by the system (1), (2).

Now from the equation (1), we have the system

$$fx + gy + hz \ldots = 0 \ldots\ldots\ldots (6).$$

$$fdx + gdy + hdz \ldots = 0$$

$$fd^{n-2}x + gd^{n-2}y + hd^{n-2}z \ldots = 0,$$

Or writing

$$X = \begin{vmatrix} y, z & \ldots \\ dy, dz & \ldots \\ \vdots & \\ d^{n-2}y, d^{n-2}z & \ldots \end{vmatrix} \ldots\ldots\ldots\ldots\ldots (7)$$

With analogous expressions for $Y, Z \ldots$ from the equations (6), $f, g, h \ldots$ are proportional to $X, Y, Z \ldots$ or eliminating by (2)

$$H(X, Y, Z \ldots) = 0 \ldots\ldots\ldots\ldots (8).$$

Conversely the integral of the equation (8) of the order $(n-2)$ is given either by the system of equations (1), (2), in which it is evident the number of arbitrary constants is reduced to $(n-2)$; or, by the equation (5), which contains in appearance $n\ (n-2)$ arbitrary constants, but which we have seen is equivalent in reality to the system (1),(2).

Thus, with three variables, the integral of

$$H(ydz - zdy,\ zdx - xdz,\ xdy - ydx) = 0 \ldots (9)$$

may be expressed in the two forms

$$H(yz_1 - y_1z,\ zx_1 - z_1x,\ xy_1 - x_1y) = 0 \ldots (10),$$

and

$$fx + gy + hz = 0 \ldots\ldots\ldots\ldots (11),$$

where

$$H(f, g, h) = 0 \ldots\ldots\ldots\ldots (12).$$

The above principles afford an elegant mode of integrating the differential equation for the lines of curvature of an ellipsoid. The equation in question is

$$(b^2 - c^2)\, xdydz + (c^2 - a^2)\, ydzdx + (a^2 - b^2)\, zdxdy = 0 \ldots (13),$$

where x, y, z are connected by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots\ldots (14);$$

writing

$$\frac{x^2}{a^2} = u,\ \frac{y^2}{b^2} = v,\ \frac{z^2}{c^2} = w \ldots\ldots\ldots (15),$$

these become

$$(b^2 - c^2)\, udvdw + (c^2 - a^2)\, vdwdu + (a^2 - b^2)\, wdudv = 0 \ldots (16),$$

$$u + v + w = 1 \ldots\ldots\ldots\ldots (17).$$

Multiplying by

$$-\{(vdu - udv)\,(wdv - vdw)\,(udw - wdu)\}^{-1},$$

the first of these becomes

$$\frac{-a^2du}{(vdu-udv)(udw-wdu)} - \frac{b^2dv}{(wdv-vdw)(vdu-udv)} - \frac{c^2dw}{(udw-wdu)(wdv-vdw)} = 0 \ldots\ldots (18);$$

but writing (17) and its derived equations under the form

$$u + (v + w) = 1 \ldots\ldots\ldots\ldots (19),$$

$$du + (dv + dw) = 0,$$

we deduce

$$-du\,(v + w) + u\,.(dv + dw) = -du \ldots\ldots (20),$$

i.e.

$$-du = -(vdu - udv) + (udw - wdu) \ldots\ldots (21).$$

and similarly

$$-dv = -(wdv - vdw) + (vdu - udv)$$

$$-dw = -(udw - wdu) + (wdv - vdw).$$

Substituting,

$$\frac{b^2-c^2}{wdv-vdw}+\frac{c^2-a^2}{udw-wdu}+\frac{a^2-b^2}{vdu-udv}=0\ldots(22);$$

the integral of which may be written under either of the forms

$$\frac{b^2-c^2}{wv_1-vw_1}+\frac{c^2-a^2}{uw_1-wu_1}+\frac{a^2-b^2}{vu_1-v_1u}=0\ldots.(23).$$

where, on account of (17),

$$u_1+v_1+w_1=1\ldots\ldots\ldots\ldots(24).$$

Or

$$fu+gv+hw=0\ldots\ldots\ldots\ldots(25),$$

f, g, h being connected by

$$\frac{b^2-c^2}{f}+\frac{c^2-a^2}{g}+\frac{a^2-b^2}{h}=0\ldots\ldots\ldots(26),$$

an equation which is satisfied identically by

$$f=\frac{b^2-c^2}{B^2-C^2},\quad g=\frac{c^2-a^2}{C^2-A^2},\quad h=\frac{a^2-b^2}{A^2-B^2}\ldots.(27).$$

Restoring x, y, z, x_1, y_1, z_1 for u, v, w, u_1, v_1, w_1, the equations to a line of curvature passing through a given point x_1, y_1, z_1, on the ellipsoid, are the equation (14) and

$$\frac{(b^2-c^2)}{a^2(y_1^2z^2-y^2z_1^2)}+\frac{(c^2-a^2)}{b^2(z_1^2x^2-z^2x_1^2)}+\frac{(a^2-b^2)}{c^2(x_1^2y^2-x^2y_1^2)}=0\ldots(28).$$

Or again, under a known form, the equation (14) and

$$\frac{(b^2-c^2)}{B^2-C^2}\cdot\frac{x^2}{a^2}+\frac{c^2-a^2}{C^2-A^2}\cdot\frac{y^2}{b^2}+\frac{a^2-b^2}{A^2-B^2}\cdot\frac{z^2}{c^2}=0\ldots(29).$$

From the equations (14), (29) it is easy to prove the well known form

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1\ldots\ldots\ldots(30);$$

in fact, multiplying (29) by m, and adding to (14), we have the equation (30), if the equations

$$\frac{1}{a^2}+m\cdot\frac{b^2-c^2}{B^2-C^2}\cdot\frac{1}{a^2}=\frac{1}{a^2+\theta}\ldots\ldots\ldots(31),$$

$$\frac{1}{b^2}+m\cdot\frac{c^2-a^2}{C^2-A^2}\cdot\frac{1}{b^2}=\frac{1}{b^2+\theta},$$

$$\frac{1}{c^2}+m\cdot\frac{a^2-b^2}{A^2-B^2}\cdot\frac{1}{c^2}=\frac{1}{(c^2+\theta)}$$

are satisfied.

But on reduction, these take the form

$$(B^2-C^2)\,\theta+(b^2-c^2)\,m\theta+ma^2(b^2-c^2)=0\ldots(32).$$
$$(C^2-A^2)\,\theta+(c^2-a^2)\,m\theta+mb^2(c^2-a^2)=0,$$
$$(A^2-B^2)\,\theta+(a^2-b^2)\,m\theta+mc^2(a^2-b^2)=0.$$

And since, by adding, an identical equation is obtained, m and θ may be determined to satisfy these equations. The values of θ, m are

$$\theta = \frac{(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)}{a^2(B^2 - C^2) + b^2(C^2 - A^2) + c^2(A^2 - B^2)} \ldots\ldots(33),$$

$$m = \frac{b^2c^2(B^2 - C^2) + c^2a^2(C^2 - A^2) + a^2b^2(A^2 - B^2)}{(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)} \ldots(34).$$

V.—NOTE ON A PROBLEM IN DYNAMICS.

In the problem of finding the trajectory of a body under the action of a central force varying inversely as the square of the distance, when the circumstances of projection are given, it is usual to employ polar co-ordinates, either r and θ or p and r. The problem may, however, be solved quite as readily and more elegantly by adhering to rectilinear co-ordinates. To shew this take the equations of motion,

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{r^3} \ldots\ldots (1), \qquad \frac{d^2y}{dt^2} = -\frac{\mu y}{r^3} \ldots\ldots (2),$$

whence, as usual, we have

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h \ldots\ldots (3).$$

Multiplying (1) by (3), we have

$$h\frac{d^2x}{dt^2} = -\frac{\mu}{r^3}\left(x^2\frac{dy}{dt} - xy\frac{dx}{dt}\right)$$

$$= -\frac{\mu}{r^2}\left(r\frac{dy}{dt} - y\frac{dr}{dt}\right) = -\mu\frac{d}{dt}\left(\frac{y}{r}\right).$$

Therefore on integration

$$-h\frac{dx}{dt} = \mu\frac{y}{r} + g \ldots\ldots (4),$$

g being an arbitrary constant.

Similarly from (2) and (3), we have

$$h\frac{dy}{dt} = \mu\frac{x}{r} + f \ldots\ldots (5).$$

Multiply (4) by y, and (5) by x, and add, then

$$\mu r + fx + gy = h^2 \ldots\ldots (6).$$

From this it appears that r the radius vector is a rational and integral function of the co-ordinates of its extremity (x and y), and therefore this is the equation to a conic section, the focus of which is at the origin.

On comparing (6) with the general polar equation to the conic section,

$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\alpha)},$$

$$\text{or} \quad r + e\cos\alpha x + e\sin\alpha y = a(1-e^2),$$

we find $$a(1-e^2) = \frac{h^2}{\mu}, \quad e^2 = \frac{f^2+g^2}{\mu^2},$$

$$a = \frac{\mu h^2}{\mu^2 - (f^2+g^2)}, \quad \tan\alpha = \frac{g}{f}.$$

To determine the velocity in the path, square and add (4) and (5), then

$$h^2v^2 = \mu^2 + \frac{2\mu}{r}(fx+gy) + a^2 + b^2,$$

But from (6) $$fx + gy = h^2 - \mu r,$$

therefore $$h^2v^2 = h^2\frac{2\mu}{r} + f^2 + g^2 - \mu^2 \ldots\ldots (7).$$

Now if V be the velocity of projection at the distance ρ, we have from (7),

$$h^2V^2 = h^2\frac{2\mu}{\rho} + f^2 + g^2 - \mu^2 \ldots\ldots (8),$$

whence, by subtracting, and dividing by h^2,

$$v^2 = V^2 + 2\mu\left(\frac{1}{r} - \frac{1}{\rho}\right) \ldots\ldots (9).$$

Again from (3), we have

$$\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right)\frac{ds}{dt} = h.$$

But if δ be the angle between ρ and the direction of projection,

$$x\frac{dy}{ds} - y\frac{dx}{ds} = \rho\sin\delta$$

at the point of projection: and as then we have $\frac{ds}{dt} = V$,

$$h = V\rho\sin\delta \ldots\ldots (10).$$

Then from (8) $$\frac{f^2+g^2-\mu^2}{h^2} = V^2 - \frac{2\mu}{\rho};$$

and therefore $$a = \frac{\mu}{\frac{2\mu}{\rho} - V^2} \ldots\ldots\ldots (11).$$

Also $$e^2 = 1 - \frac{V^2\rho^2\sin^2\delta}{\mu^2}\left(\frac{2\mu}{\rho} - V^2\right) \ldots (12).$$

As the species of conic section depends on whether e is less, equal to, or greater than unity, it appears that the path is an ellipse, a parabola, or a hyperbola, according as

$$\frac{2\mu}{\rho} > = \text{or} < V^2,$$

and therefore the species of conic section is independent of the angle of projection.

To find the angle α, we have

$$\tan \alpha = \frac{g}{f}, \quad \text{or} \cos \alpha = \frac{f}{\sqrt{(f^2+g^2)}} = \frac{f}{\mu e}.$$

Now supposing the direction of projection to coincide with the axis of x, we have $x = \rho$ when $y = 0$, hence (6) gives us

$$f = \frac{h^2}{\rho} - \mu = V^2\rho \sin^2 \delta - \mu;$$

and therefore

$$\cos \alpha = \frac{V^2\rho \sin^2 \delta - \mu}{\mu e} = \frac{V^2\rho \sin^2 \delta - \mu}{\left\{\mu^2 - V^2\rho^2 \sin^2 \delta \left(\frac{2\mu}{\rho} - V^2\right)\right\}^{\frac{1}{2}}}.$$

If we combine (8) and (11) we get, as an expression for the velocity,

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}.$$

G.

VI.—ON THE INTEGRATION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS BY THE METHODS OF MONGE AND LAGRANGE.

By W. Walton, M.A. Trinity College.

In Professor De Morgan's valuable work on the Differential and Integral Calculus (p. 722), the method of Monge for obtaining the first integrals of a partial differential equation of the second order and first degree, is objected to as probably from the nature of the case leading to results of insufficient generality. This method commences with assuming the coefficient of s, and consequently the left-hand member in the equation

$$Pdy\,dp + Rdx\,dq - Sdx\,dy = s\,(Pdy^2 - Qdx\,dy + Rdx^2),$$

or, as it may be written, $\sigma = s\alpha$, each equal to zero. The Professor observes that "in the equation $\sigma = s\alpha$ we have begun by presuming the existence of a solution which allows α to vanish, when of course σ vanishes. The solution we thus obtain may be the most general of its kind,—that is, of those which allow α and σ to vanish; but how do we ascertain that there are no solutions in which this is impossible? or how do

we know that there are not some in which when a vanishes, s necessarily becomes infinite?" The generality of Lagrange's integration of partial differential equations of the first order, the Professor establishes by an à posteriori demonstration. The object of the paper which we here lay before the readers of the Journal, is to develop a fundamental idea which characterizes the processes of integration given by Monge and Lagrange, whatever be the order of the equation or the number of the variables, and to shew by à priori considerations that the subsidiary assumptions do not involve any sacrifice of generality whatever.

A. We shall commence with the discussion of partial differential equations of the first order between three variables x, y, z. The general form of this class of equations is

$$Pp + Qq = R \ldots\ldots\ldots\ldots (1),$$

where P, Q, R, are given functions of x, y, z, and p, q, represent $\frac{dz}{dx}, \frac{dz}{dy}$, the partial differential coefficients of z.

Since z is a function of x and y, we have

$$dz = pdx + qdy \ldots\ldots\ldots\ldots (2).$$

Eliminating q between (1) and (2) we get

$$(Pdy - Qdx)p = Rdy - Qdz \ldots\ldots (3).$$

Now since the two ratios subsisting between the three differentials dx, dy, dz, are subject to only a single equation (2), it is evident that we may establish any relation whatever between either of these ratios and the variables x, y, z, without restricting in any degree the absolute or relative values of these variables, and therefore without limiting the generality of the equation (1): assume then

$$Pdy - Qdx = 0 \ldots\ldots\ldots\ldots (4),$$

and therefore, by (3),

$$Rdy - Qdz = 0 \ldots\ldots\ldots\ldots (5).$$

Suppose now that from these two equations, (4) and (5), we can obtain two perfect differentials

$$dF = 0, \quad dG = 0,$$

where F, G, are certain functions of x, y, z. Since these two equations are simultaneous, or, which is the same thing, since $dF = 0$ whenever $dG = 0$, and vice versâ, it is evident that F and G must be functional of each other.

Hence we see that from an assumption involving no sacrifice of generality we obtain

$$\phi(F, G) = 0,$$

which is consequently a relation among the three variables x, y, z, coextensive in generality with the equation (1), and therefore the complete integral of the proposed equation. The symbol ϕ evidently denotes undefined and therefore arbitrary functionality.

The nature of the assumption on which this method of integration hinges may receive a convenient illustration from the algebraic geometry of surfaces. Let x, y, z, be the co-ordinates of any point in a surface, and let θ denote the angle between the axis of x and the projection upon the plane of x, y, of an elemental arc between this point and a contiguous point $x + dx,\ y + dy,\ z + dz$; then, the number of contiguous points being infinite, it is clear that we are at liberty to pass to a new point such that $\tan\theta\left(=\frac{dy}{dx}\right) = \frac{Q}{P}$: this is admissible at whatever point x, y, z, we choose to start, and does not presume any peculiarity in the form of the surface. For the success of Lagrange's integration, it is necessary that from the two auxiliary equations (4) and (5) it be possible to obtain an equation containing only two variables, the differentials of which it involves. Probably when this is not possible, the auxiliary equations may correspond to a discontinuous line traced on the surface.

B. The general form of a partial differential equation of the second order and first degree between three variables is

$$Pr + Qs + Rt = S \ldots\ldots\ldots\ldots (1),$$

where P, Q, R, S, designate assigned functions of x, y, z, p, q, and where r, s, t, denote respectively the partial differential coefficients

$$\frac{d^2z}{dx^2},\ \frac{d^2z}{dx\,dy},\ \frac{d^2z}{dy^2}.$$

By the principles of differentiation we have

$$dz = pdx + qdy \ldots\ldots\ldots\ldots (2),$$
$$dp = rdx + sdy \ldots\ldots\ldots\ldots (3),$$
$$dq = sdx + tdy \ldots\ldots\ldots\ldots (4).$$

Eliminating r and t between the three equations (1), (3), (4), we shall get

$$Pdy\,dp + Rdx\,dq - Sdx\,dy = s\,(Pdy^2 - Qdx\,dy + Rdx^2) \ldots (5).$$

But since the four ratios subsisting between the five differentials dx, dy, dz, dp, dq, are subject only to three equations (2), (3), (4), it is evident that we may establish any relation whatever between any one of these ratios and the variables x, y, z, p, q, without restricting in any degree the absolute or

relative values of x, y, z, p, q, r, s, t, and therefore without violating the generality of the proposed equation (1). Assume then

$$Pdy^2 - Qdx\,dy + Rdx^2 = 0 \ldots\ldots\ldots (6);$$

whence, by (5), we have

$$Pdy\,dp + Rdx\,dq = Sdx\,dy \ldots\ldots (7).$$

Now, from (6), by the solution of a quadratic, we shall have generally two values of $\frac{dy}{dx}$. Let k', k'', denote the two values, which will generally be functions of x, y, z, p, q. Then, from (6) and (7), we shall have the two following systems of simultaneous equations,

$$\left.\begin{aligned} dy &= k'dx, \\ Pk'dp + Rdq &= Sk'dx \end{aligned}\right\} \ldots\ldots\ldots\ldots (8);$$

$$\left.\begin{aligned} dy &= k''dx, \\ Pk''dp + Rdq &= Sk''dx. \end{aligned}\right\} \ldots\ldots\ldots\ldots (9).$$

Suppose that from (8) we can obtain two perfect differentials

$$dF = 0, \quad dG = 0,$$

where F, G, are certain functions of x, y, z, p, q. Then, by the reasoning employed in the integration of the equation of the first order, we shall have, as a first integral coextensive in generality with the proposed equation (1),

$$\phi(F, G) = 0 \ldots\ldots\ldots\ldots (10),$$

where the symbol ϕ denotes arbitrary functionality.

In the same way from the system (9) we may get

$$\phi_1(F_1, G_1) = 0 \ldots\ldots\ldots\ldots (11),$$

ϕ_1, F_1, G_1, corresponding to ϕ, F, G.

The integration of either of the equations (10) and (11) will give rise to an equation in x, y, z, which will be the same in both cases, as will be evident when it is considered that each of them is as general as the equation (1).

If it be more convenient, which is generally the case, we may combine the equations (10) and (11) as simultaneous, in order to obtain the complete integral. The admissibility of this arises from the fact that the complete integral must be such as to satisfy each of them singly, and therefore any equations resulting from their combination; the converse proposition being equally clear, viz. that any relation between x, y, z, which shall satisfy any two equations deduced from (10) and (11), shall likewise satisfy the equations (10) and (11) themselves.

We have not yet taken into consideration the difficulty started by Professor De Morgan regarding the possibility of

s becoming infinite when a is assumed equal to zero. Since however s is some function of x, y, z, p, q, and since the assumption $a = 0$, as we have shewn, does not in any way restrict the relations existing among these quantites, the difficulty ceases. It is not necessary to dwell on the cases in which the method of Monge fails, the ordinary treatises being sufficiently explicit on these points. We will proceed now to equations of the first order involving any number of variables.

C. The general form of a partial differential equation of the first order involving any number of variables $x_1, x_2, x_3, \ldots x_n, u$, is, u being reckoned the dependent variable,

$$P_1p_1 + P_2p_2 + P_3p_3 + \ldots + P_np_n = Q \ldots (1),$$

where $P_1, P_2, P_3, \ldots P_n$, are proposed functions of $x_1, x_2, x_3, \ldots x_n, u$, and $p_1, p_2, p_3, \ldots p_n$, denote respectively the partial differential coefficients

$$\frac{du}{dx_1}, \frac{du}{dx_2}, \frac{du}{dx_3}, \ldots \frac{du}{dx_n}.$$

Now we have

$$du = p_1dx_1 + p_2dx_2 + p_3dx_3 + \ldots + p_n dx_n \ldots (2);$$

hence, from (1), eliminating p_1,

$$P_1du - Qdx_1 + (P_2dx_1 - P_1dx_2)p_2 + (P_3dx_1 - P_1dx_3)p_3 + \ldots + (P_ndx_1 - P_1dx_n)p_n = 0 \ldots\ldots (3).$$

Now since the equation (2) involves, beside the quantities $p_1, p_2, p_3, \ldots p_n$, the n ratios subsisting between the $n + 1$ differentials $dx_1, dx_2, dx_3, \ldots dx_n, du$, it is clear that we may establish arbitrarily between each of any $n - 1$ of the ratios and the variables $x_1, x_2, x_3, \ldots x_n, u$, any relation we please without in any degree limiting the absolute or relative values of $x_1, x_2, x_3, \ldots x_n, u, p_1, p_2, p_3, \ldots p_n$, and therefore it is manifest that the assumption of these relations will in no wise restrict the generality of the proposed equation (1). Assume then

$$P_2dx_1 - P_1dx_2 = 0, \quad P_3dx_1 - P_1dx_3 = 0, \ldots P_ndx_1 - P_1dx_n = 0,$$

and therefore from (3) we have also

$$P_1du - Qdx_1 = 0.$$

Suppose that from these n simultaneous equations we can obtain n perfect differentials

$$dF_1 = 0, \quad dF_2 = 0, \quad dF_3 = 0 \ldots dF_n = 0.$$

These equations shew that our auxiliary assumptions, which we have shewn to introduce no restriction among the variables of the proposed equation (1), are equivalent to assuming any

one of the functions $F_1, F_2, F_3, \ldots F_n$, to be an undefined or arbitrary function of all the rest; hence the complete integral of (1) will be

$$\phi(F_1, F_2, F_3, \ldots F_n) = 0,$$

ϕ denoting arbitrary functionality.

D. The general form of a partial differential equation of the n^{th} order and of the first degree between three variables x, y, z, is

$$\overset{n}{u}_{0}\,.\,\overset{n}{z}_{0} + \overset{n-1}{u}_{1}\,.\,\overset{n-1}{z}_{1} + \overset{n-2}{u}_{2}\,.\,\overset{n-2}{z}_{2} + \ldots + \overset{1}{u}_{n-1}\,.\,\overset{1}{z}_{n-1} + \overset{0}{u}_{n}\,.\,\overset{0}{z}_{n} = v \ldots (1),$$

$\overset{n-r}{z}_{r}$ denoting $\dfrac{d^n z}{dx^{n-r}dy^r}$ and $\overset{n-r}{u}_{r}$ as well as v representing some function of x, y, z, and the partial differential coefficients of orders inferior to the n^{th}.

Then, corresponding to the equations (A, 3) and (A, 4), we shall have

$$\begin{aligned}
d\overset{n-1}{z}_{0} &= \overset{n}{z}_{0}\,dx + \overset{n-1}{z}_{1}\,dy,\\
d\overset{n-2}{z}_{1} &= \overset{n-1}{z}_{1}\,dx + \overset{n-2}{z}_{2}\,dy,\\
d\overset{n-3}{z}_{2} &= \overset{n-2}{z}_{2}\,dx + \overset{n-3}{z}_{3}\,dy,\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
d\overset{0}{z}_{n-1} &= \overset{1}{z}_{n-1}\,dx + \overset{0}{z}_{n}\,dy;
\end{aligned}$$

hence from (1) we get

$$\overset{0}{z}_{n}\left\{\overset{0}{u}_{n}\,dx^n - \overset{1}{u}_{n-1}\,dx^{n-1}dy + \overset{2}{u}_{n-2}\,dx^{n-2}dy^2 - \ldots + (-)^n\overset{n}{u}_{0}\,dy^n\right\} = w,$$

where w represents an expression involving no partial differential coefficient of z of an order higher than the $(n-1)^{\text{th}}$, being a function of these coefficients and of their total differentials as well as of x, y, z.

Then, corresponding to the relations (6) and (7) of (B), we shall have

$$\overset{0}{u}_{n}\,.\,dx^n - \overset{1}{u}_{n-1}\,dx^{n-1}dy + \overset{2}{u}_{n-2}\,dx^{n-2}dy^2 - \ldots + (-)^n\overset{n}{u}_{0}\,dy^n = 0 \ldots (2),$$

$$w = 0 \ldots\ldots\ldots\ldots\ldots (3).$$

From (2), an equation of the n^{th} degree, we can obtain n values of $\dfrac{dy}{dx}$, and therefore, corresponding to the two systems

of simultaneous equations (B, 8) and (B, 9), we shall have n such systems; and the remarks which we made respecting the pair of systems and their integrals, are evidently applicable mutatis mutandis to the more general case of the n systems.

E. We have in the preceding pages discussed the solutions of partial differential equations of all orders for three variables, and of the first order for any number of variables. It is unnecessary to extend this paper to greater length by the consideration of the general case of an equation of any number of variables and of any order, the principles which have been developed above being evidently universally applicable.

VII.—ON THE ATTRACTIONS OF CONDUCTING AND NON-CONDUCTING ELECTRIFIED BODIES.

In measuring the action exerted upon an electrified body, by a quantity of free electricity distributed in any manner over another body, the methods followed in the cases in which the attracted body is conducting and non-conducting are different. Now, the only difference between the state of a conducting body and that of a non-conducting body is, that the electricity is held upon a conducting body by the pressure of the atmosphere (to a certain extent at least), while on a non-conducting body it is held by the *friction* of the particles of the body.

To find the attraction of an electrical mass E, on a non-conducting electrified body A, the obvious way is to proceed as in ordinary cases of attraction, considering the electricity on A as the attracted mass.

In finding the action on a conducting body A, the method followed is to consider its electricity as exerting no pressure upon the particles of the body, but to disturb its equilibrium, by making the pressure of the air unequal at different parts of its surface. These two methods of measuring the action of E on A *should* obviously lead to the same result, since the action must be the same, whether A be conducting or non-conducting, the distribution remaining the same. It is the object of the following paper to show that they *do* lead to the same result.

We must first find the pressure of an element of the electricity of A, on the atmosphere.

Let ds be the area of the element, and ρds its electrical mass. Let ds form part of another element σ, indefinitely larger than ds in every direction, but so small that it may be considered as plane. Now, if $\rho\sigma$ be a material plane, it can exercise no attraction on ρds, in a direction perpendicular to

the plane, and it may be readily shown that this is also true if $\rho\sigma$ be a plate of matter of different densities, arranged in parallel planes, the thickness being either finite or indefinitely small, and the law of density being any whatever.

Hence, the force acting on ρds is due to the repulsion of all the electrical mass, except σ; and, since the electricity on A is in equilibrium under the influence of E, the repulsion acts along the normal through ds, and is in magnitude $2\pi\rho^2 ds$, (see vol. III. p. 75), which is therefore the pressure of ds on the air. Hence, if p be the barometric pressure of the atmosphere, the pressure on ds, perpendicular to the surface, is

$$(p - 2\pi\rho^2)\,ds.$$

Hence, if X be the whole pressure on A, resolved along a fixed line $X'X$, and if ν be the angle which the normal through ds makes with this line, we have

$$X = -\int\int(p - 2\pi\rho^2)\cos\nu\, ds,$$

the integrals being extended over the surface of A. Now,

$$\int\int p\cos\nu\, ds = 0,$$

since the pressure of the atmosphere does not disturb the equilibrium of A. Hence, we have

$$X = 2\pi\int\int\rho^2\cos\nu\, ds. \ldots\ldots\ldots\ldots\ldots (a),$$

which is the expression for the attraction on a conducting body A, either separate from the body on which E is distributed, or connected with it.

To show that this is identical with the expression for the attraction of E on the electricity of A, let $R\rho ds$ and $R'\rho ds$ be the components of the repulsion on ρds, which are due to E, and to the electricity of A; and let α, α' be the angles which their directions make with XX'. Then we shall have

$$2\pi\rho\cos\nu = R\cos\alpha + R'\cos\alpha';$$

therefore

$$X = \int\int(R\cos\alpha + R'\cos\alpha')\,ds.$$

Now, $\int\int R'\cos\alpha' ds$ is the attraction of the electricity of A on itself in the direction XX', and is therefore = 0. Hence,

$$X = \int\int R\cos\alpha ds. \ldots\ldots\ldots\ldots\ldots (b).$$

But this expression for X is the attraction of E on the electricity of A, and hence the two methods of measuring the action lead to the same result.

P. Q. R.

VIII.—REMARKS ON A THEOREM OF M. CATALAN.

By George Boole.

M. Catalan, in *Liouville's Journal*, vol. VI. p. 81, pursuing the consequences of the celebrated theorem of Dirichlet, has arrived at the following result: *If V be the value of the definite multiple integral*,

$$\iint \ldots \frac{dx_1 dx_2 \ldots dx_{n-1}}{x_n} f(a_1 x_1 + a_2 x_2 \ldots + a_n x_n) \ldots . (1);$$

where $x_n^2 = 1 - x_1^2 - x_2^2 \ldots - x_{n-1}^2$, *and the integrations are extended to all positive values of the* $n-1$ *independent variables* $x_1, x_2, \ldots x_{n-1}$, *subject to the condition*

$$x_1^2 + x_2^2 \ldots + x_{n-1}^2 < 1;$$

then

$$V = \frac{\pi^{\frac{n-1}{2}}}{2^{n-1}\Gamma\left(\frac{n-1}{2}\right)} \int_0^{\frac{1}{2}\pi} d\theta \, (\sin\theta)^{n-2} f(A \cos\theta) \ldots . (2).$$

From this result, which I shall prove to be erroneous, the author incorrectly deduces, as a particular consequence, the theorem of Poisson,

$$\int_0^{\pi}\int_0^{2\pi} du dv \sin u \, f(a_1 \cos u + a_2 \sin u \cos v + a_3 \sin u \sin v)$$
$$= 2\pi \int_0^{\pi} d\theta \sin\theta f(A \cos\theta) \ldots . . (3);$$

well known as the foundation of his solution of the partial differential equation

$$\frac{d^2u}{dt^2} = a^2\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right),$$

which occupies so conspicuous a place in the mathematical theories of Heat, Attraction, and Electricity. On account of the great importance of this application, and of the interest attaching to the subject of definite integrals generally, I purpose here to shew in what respects M. Catalan's demonstration is unsound, and to investigate the true form of the theorem of which Poisson's is a particular case.

M. Catalan assumes a new set of n variables $u_1, u_2, \ldots u_n$, connected with $x_1, x_2, \ldots x_n$, by linear equations, of which one is

$$a_1 x_1 + a_2 x_2 \ldots + a_n x_n = A u_1,$$

where $A = \sqrt{(a_1^2 + a_2^2 \ldots + a_n^2)}$, and the rest are so determined as to satisfy the condition

$$x_1^2 + x_2^2 \ldots + x_n^2 = u_1^2 + u_2^2 \ldots + u_n^2.$$

He thus obtains for the transformed integral,

$$V = \iint \ldots \frac{du_1 du_2 \ldots du_{n-1}}{u_n} f(Au_1) \ldots\ldots\ldots (4),$$

which he assumes to extend to all positive values of $u_1, u_2 \ldots u_{n-1}$, satisfying the condition

$$u_1^2 + u_2^2 \ldots + u_{n-1}^2 \leqq 1 \ldots\ldots\ldots\ldots (5).$$

Now, neither of the particulars involved in this assumption is realized. The equation of the limits is not

$$u_1^2 + u_2^2 \ldots + u_{n-1}^2 \leqq 1;$$

neither do the positive values of $x_1, x_2, \ldots x_{n-1}$, invariably correspond to positive values of $u_1, u_2, \ldots u_{n-1}$. The erroneous character of M. Catalan's conclusions will be more distinctly seen, if we assume a particular form of f, and value of n, and calculate the two values of V above quoted. Let $n = 3$, and $f(t) = t$, then by the above transformation we should have

$$\iint \frac{dx_1 dx_2}{\sqrt{(1 - x_1^2 - x_2^2)}} \{a_1 x_1 + a_2 x_2 + a_3 \sqrt{(1 - x_1^2 - x_2^2)}\}$$

$$= \iint \frac{du_1 du_2}{\sqrt{(1 - u_1^2 - u_2^2)}} \{A\sqrt{(1 - u_1^2 - u_2^2)}\} = A \iint du_1 du_2 \ldots . (6);$$

the integrations extending to all positive values of the variables satisfying the conditions $x_1^2 + x_2^2 \leqq 1$, $u_1^2 + u_2^2 \leqq 1$. Now, by Liouville's extension of Dirichlet's theorem, if U be the value of the definite integral,

$$\iint \ldots dx_1 dx_2 \ldots x^{a-1} y^{b-1} \ldots f\left\{\left(\frac{x}{a}\right)^p + \left(\frac{y}{\beta}\right)^q + \&c.\right\},$$

taken through the positive limits of the inequality,

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{\beta}\right)^q + \&c. \leqq 1;$$

$$\text{then } U = \frac{a^a \beta^b \ldots}{pq \ldots} \frac{\Gamma\left(\frac{a}{p}\right) \Gamma\left(\frac{b}{q}\right) \ldots}{\Gamma\left(\frac{a}{p} + \frac{b}{q} + \ldots\right)} \int_0 du f(u) u^{\frac{a}{p} + \frac{b}{q} \ldots - 1} \ldots . (7).$$

Applying this theorem, we get for the first member of (6)

$$V = \frac{a_1 + a_2}{4} \frac{\Gamma(1) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \int_0 du\, u^{\frac{1}{2}} (1 - u)^{-\frac{1}{2}} + \frac{a_3 \pi}{4}$$

$$= \frac{a_1 + a_2 + a_3}{4} \pi,$$

since $\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}}$, and $\int_0^1 du\, u^{\frac{1}{2}} (1-u)^{-\frac{1}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)} = \frac{\pi}{2}$.

But from the second member of (6), as from the second member of (2), we have

$$V = \frac{\sqrt{(a_1^2 + a_2^2 + a_3^2)}}{4}\pi;$$

and these results cannot be made to correspond, except for particular values of a_1, a_2, a_3, which are supposed to be entirely arbitrary.

I shall now develop what I conceive to be the proper mode of investigation.

Let $f'(t) = \frac{df(t)}{dt}$, and let us consider the definite multiple integral

$$\int\int \ldots dx_1 dx_2 \ldots dx_n f'(a_1x_1 + a_2x_2 \ldots a_nx_n) = V \ldots (8),$$

the integrations extending to all real values, positive or negative, of the n independent variables, satisfying the condition

$$x_1^2 + x_2^2 \ldots + x_n^2 \leqq 1 \ldots\ldots (9).$$

Let the system $x_1, x_2 \ldots x_n$, be linearly transformed into the system $u_1, u_2 \ldots u_n$, in accordance with the conditions

$$\left.\begin{aligned} a_1x_1 + a_2x_2 \ldots + a_nx_n &= Au_n, \\ x_1^2 + x_2^2 \ \ldots + \ x_n^2 &= u_1^2 + u_2^2 \ldots + u_n^2, \end{aligned}\right\} \ldots\ldots (10).$$

We shall obviously have $A = \sqrt{(a_1^2 + a_2^2 \ldots + a_n^2)}$, and $dx_1 dx_2 \ldots dx_n = du_1 du_2 \ldots du_n$; whence

$$V = \int\int \ldots du_1 du_2 \ldots du_n f'(Au_n) \ldots\ldots (11),$$

the integrations extending to all real values of $u_1, u_2 \ldots u_n$, subject to the condition

$$u_1^2 + u_2^2 \ldots + u_n^2 \leqq 1 \ldots\ldots\ldots (12).$$

Integrating with respect to u_n between the limits

$$u_n = -\sqrt{(1 - u_1^2 \ldots - u_{n-1}^2)} \text{ and } u_n = \sqrt{(1 - u_1^2 \ldots - u_{n-1}^2)},$$

we have

$$V = \int\int \ldots du_1 du_2 \ldots du_{n-1} \frac{\left\{f\left(A\sqrt{1-u_1^2 \ldots - u_{n-1}^2}\right) - f\left(-A\sqrt{1-u_1^2 \ldots - u_{n-1}^2}\right)\right\}}{A} \ldots\ldots (13);$$

the equation of the limits now becoming

$$u_1^2 + u_2^2 \ldots + u_{n-1}^2 \leqq 1 \ldots\ldots\ldots (14).$$

Now, by Liouville's theorem (7)

$$\int\int \ldots du_1 du_2 \ldots du_{n-1} f\left\{A\sqrt{(1 - u_1^2 \ldots - u_{n-1}^2)}\right\} = \frac{1}{2^{n-1}} \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^{n-1}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^1 dv\, v^{\frac{n-3}{2}} f\left\{A\sqrt{(1-v)}\right\},$$

$$\iint \ldots du_1\, du_2 \ldots du_{n-1} f\{-A\sqrt{(1-u_1^2 \ldots -u_{n-1}^2)}\}$$
$$= \frac{1}{2^{n-1}} \frac{\{\Gamma(\frac{1}{2})\}^{n-1}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^1 dv\, v^{\frac{n-3}{2}} f\{-A\sqrt{(1-v)}\};$$

the integrations being extended through the positive values of $u_1, u_2 \ldots u_{n-1}$ only. But $u_1, u_2 \ldots u_{n-1}$ enter under the sign of integration in even powers. Hence the value of V will be found by substituting the above expressions in the second member of (13), and multiplying the result by 2^{n-1}, which will extend the integrations to all real values of $u_1, u_2 \ldots u_{n-1}$. We thus find

$$V = \frac{\{\Gamma(\frac{1}{2})\}^{n-1}}{A\Gamma\left(\frac{n-1}{2}\right)} \int_0^1 dv\, v^{\frac{n-3}{2}} [f\{A\sqrt{(1-v)}\} - f\{-A\sqrt{(1-v)}\}] \ldots (14).$$

Let $v = (\sin\theta)^2$, then $dv = 2\sin\theta\cos\theta d\theta$, $\sqrt{(1-v)} = \cos\theta$, also $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$, therefore

$$V = \frac{2\pi^{\frac{n-1}{2}}}{A\Gamma\left(\frac{n-1}{2}\right)} \left\{ \int_0^{\frac{1}{2}\pi} d\theta\, (\sin\theta)^{n-2} \cos\theta f(A\cos\theta) - \int_0^{\frac{1}{2}\pi} d\theta\, (\sin\theta)^{n-2} \cos\theta f(-A\cos\theta) \right\}.$$

In the second integral, let $\theta = \pi - \theta'$, then $d\theta = -d\theta'$, $\cos\theta = -\cos\theta'$, $\sin\theta = \sin\theta'$, the limits of θ' being π and $\frac{1}{2}\pi$, hence

$$-\int_0^{\frac{1}{2}\pi} d\theta (\sin\theta)^{n-2} \cos\theta f(-A\cos\theta) = -\int_{\pi}^{\frac{1}{2}\pi} d\theta' \sin\theta'^{n-2} \cos\theta' f(A\cos\theta')$$
$$= \int_{\frac{1}{2}\pi}^{\pi} d\theta' \sin\theta'^{n-2} \cos\theta' f(A'\cos\theta').$$

Taking away the accent from θ', and adding to the first integral in the value of V, we find

$$V = \frac{2\pi^{\frac{n-1}{2}}}{A\Gamma\left(\frac{n-1}{2}\right)} \int_0^{\pi} d\theta\, (\sin\theta)^{n-2} \cos\theta f(A\cos\theta) \ldots\ldots (15).$$

Now resuming (8), integrate with respect to x_n between the limits $x_n = -\sqrt{(1-x_1^2 \ldots -x_{n-1}^2)}$ and $x_n = \sqrt{(1-x_1^2 \ldots -x_{n-1}^2)}$, we have

$$V = \iint \ldots \frac{dx_1 dx_2 \ldots dx_{n-1}}{a_n} \left\{ \begin{matrix} f\{a_1 x_1 \ldots + a_{n-1}x_{n-1} + a_n\sqrt{(1-x_1^2 \ldots -x_{n-1}^2)}\} \\ -f\{a_1 x_1 \ldots + a_{n-1}x_{n-1} - a_n\sqrt{(1-x_1^2 \ldots -x_{n-1}^2)}\} \end{matrix} \right\}$$
$$\ldots\ldots (16),$$

the integrations now extending to all real values of $x_1, x_2 \ldots x_{n-1}$, satisfying the condition

$$x_1^2 + x_2^2 \ldots + x_{n-1}^2 \leqq 1.$$

Comparing this result with (15), and writing V in the place of $a_n V$, we have the following general theorem.

THEOREM. *If V be the value of the definite multiple integral,*

$$\int\int \ldots dx_1 dx_2 \ldots dx_{n-1} \{f(a_1 x_1 \ldots + a_{n-1}x_{n-1} + a_n x_n) - f(a_1 x_1 \ldots + a_{n-1}x_{n-1} - a_n x_n)\},$$

extending to all real values of $x_1, x_2 \ldots x_{n-1}$ and to all real and positive values of x_n, which satisfy the conditions

$$x_1^2 + x_2^2 \ldots + x_{n-1}^2 \leqq 1,$$

$$x_1^2 + x_2^2 \ldots + x_n^2 = 1,$$

then

$$V = \frac{2a_n \pi^{\frac{n-1}{2}}}{A\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi d\theta \, (\sin\theta)^{n-2} \cos\theta \, f(A \cos\theta) \ldots (17).$$

In the case of $n = 3$, I shall now shew that this may be resolved into a beautiful system of four allied theorems, of which that of Poisson is one.

We have first to consider the expression

$$\int\int dx_1 dx_2 \{f(a_1x_1 + a_2x_2 + a_3x_3) - f(a_1x_1 + a_2x_2 - a_3x_3)\} \ldots (18).$$

Let $x_1 = \cos u$, $x_2 = \sin u \cos v$, $x_3 = \sin u \sin v$. Transforming in the usual way, $dx_1 dx_2 = (\sin u)^2 \sin v \, du \, dv$. To find the limits we may reason thus: The variables x_1, x_2, must each, independently of the sign of the other, admit of all values from -1 to 1, and the dependent variable x_3, of all values from 0 to 1. This can only be effected by assigning to both u and v the limits 0 and π; whence

$$V = \int_0^\pi \int_0^\pi du \, dv \, (\sin u)^2 \sin v \, f(a_1 \cos u + a_2 \sin u \cos v + a_3 \sin u \sin v)$$

$$- \int_0^\pi \int_0^\pi du \, dv \, (\sin u)^2 \sin v \, f(a_1 \cos u + a_2 \sin u \cos v - a_3 \sin u \sin v).$$

In the second integral put $v = 2\pi - v'$, and transforming, we get

$$- \int_0^\pi \int_{2\pi}^\pi du \, dv' (\sin u)^2 \sin v' \, f(a_1 \cos u + a_2 \sin u \cos v' + a_3 \sin u \sin v')$$

$$= \int_0^\pi \int_\pi^{2\pi} du \, dv' (\sin u)^2 \sin v' \, f(a_1 \cos u + a_2 \sin u \cos v' + a_3 \sin u \sin v')$$

which, on omitting the accent, and connecting the result with the first integral in the value of V, will give

$$V=\int_0^{\pi}\int_0^{2\pi} du\,dv(\sin u)^2\sin v f(a_1\cos u+a_2\sin u\cos v+a_3\sin u\sin v).$$

Let $a_1\cos u + a_2\sin u\cos v + a_3\sin u\sin v = U$, then comparing the last result with (17),

$$\int_0^{\pi}\int_0^{2\pi} du dv(\sin u)^2\sin v f(U)=\frac{2a_3\pi}{A}\int_0^{\pi} d\theta\sin\theta\cos\theta f(A\cos\theta)\ldots\text{(I)},$$

which is the first of the four theorems above alluded to.

If we now take x_2 x_3 as the independent variables, subject to the condition, $x_2^2 + x_3^2 = 1$, and determine x_1 by the equation $x_1^2 + x_2^2 + x_3^2 = 1$; we shall obviously have, by inspection of (17),

$$\iint dx_2\,dx_3\,\{f(a_2x_2 + a_3x_3 + a_1x_1) - f(a_2x_2 + a_3x_3 - a_1x_1)\}$$
$$=\frac{2a_1\pi}{A}\int_0^{\pi} d\theta\sin\theta\cos\theta f(A\cos\theta).$$

From the values of x_2, x_3, above assumed, we have

$$dx_2\,dx_3 = \sin u\cos u\,du\,dv.$$

Here x_1 is to be positive, while x_2, x_3 independently admit all values from -1 to $+1$. These conditions give for the limits of u, 0 and $\frac{1}{2}\pi$, for those of v, 0 and 2π. Hence

$$V=\int_0^{\frac{1}{2}\pi}\int_0^{2\pi} du\,dv\sin u\cos u f(a_2\sin u\cos v+a_3\sin u\sin v+a_1\cos u),$$
$$-\int_0^{\frac{1}{2}\pi}\int_0^{2\pi} du\,dv\sin u\cos u f(a_2\sin u\cos v+a_3\sin u\sin v-a_1\cos u).$$

In the second integral let $u = \pi - u'$; and proceeding as before, we finally get

$$\int_0^{\pi}\int_0^{2\pi} du\,dv\sin u\cos u f(U)=\frac{2a_1\pi}{A}\int_0^{\pi} d\theta\sin\theta\cos\theta f(A\cos\theta)$$
$$\ldots\text{(II)},$$

which is the second theorem of the system.

Next, let x_3, x_1 be the independent variables, x_2 being given by the condition $x_1^2 + x_2^2 + x_3^2 = 1$. Proceeding as in the two last cases, we have for our third theorem

$$\int_0^{\pi}\int_0^{2\pi} du\,dv(\sin u)^2\cos v f(U)=\frac{2a_2\pi}{A}\int_0^{\pi} d\theta\sin\theta\cos\theta f(A\cos\theta)$$
$$\ldots\text{(III)}.$$

Now multiply (II) by a_1, (III) by a_2, (I) by a_3, and add the results, we have

$$\int_0^{\pi}\int_0^{2\pi} du dv\sin u(a_1\cos u + a_2\sin u\cos v + a_3\sin u\sin v)f(U)$$
$$=\frac{2(a_1^2+a_2^2+a_3^2)\pi}{A}\int_0^{\pi} d\theta\sin\theta\cos\theta f(A\cos\theta);$$

or $\int_0^{\pi}\int_0^{2\pi} du dv \sin u\, Uf(U) = 2A\pi\int_0^{\pi} d\theta \sin\theta \cos\theta f(A\cos\theta)$.

Now the function f being arbitrary, for $Uf(U)$ write $f(U)$, and for $A\cos\theta f(A\cos\theta)$ write $f(A\cos\theta)$, then

$$\int_0^{\pi}\int_0^{2\pi} du dv \sin u f(U) = 2\pi\int_0^{\pi} d\theta \sin\theta f(A\cos\theta). \ldots\ldots (\mathrm{IV});$$

which is Poisson's theorem, the last of the allied system.

It is worthy of observation that I, II, III, may be deduced from IV, by differentiating with respect to the constants.

The following result, of which and of some others the demonstration is reserved for a future occasion, constitutes the generalization of IV, and is deduced from the same general formula of definite integration (17). viz

$$\int_0^{\pi}\int_0^{\pi}\ldots\int_0^{2\pi} d\theta_1 d\theta_2 \ldots d\theta_n U_n f\{\Sigma_{r-1}^{r=n+1}(a_r V_r)\} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\int_0^{\pi} d\theta (\sin\theta)^{n-1} f(A\cos\theta),$$

wherein $U_n = (\sin\theta_1)^{n-1}(\sin\theta_2)^{n-2}\ldots(\sin\theta_{n-1})$,

and $V_r = \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{r-1}\cos\theta_r$, up to V_n inclusive,

then $V_{n+1} = \sin\theta_1 \sin\theta_2 \ldots \sin\theta_n$;

moreover $A = (a_1^2 + a_2^2 \ldots + a_{n+1}^2)^{\frac{1}{2}}$.

Lincoln, January 1843.

IX.—ON LAGRANGE'S THEOREM.

By Arthur Cayley, B.A. Fellow of Trinity College.

1. The value given by Lagrange's theorem for the expansion of any function of the quantity (x), determined by the equation

$$x = u + hfx \ldots\ldots\ldots (1),$$

admits of being expressed in rather a remarkable symbolical form. The *à priori* deduction of this, independently of any expansion, presents some difficulties; I shall therefore content myself with showing that the form in question satisfies the equations

$$\frac{d}{du}\cdot\int F'x\, fx\, dx = \frac{d}{dh}\cdot\int F'x\, dx \ldots (2),$$

$$Fx = Fu \text{ for } h = 0 \ldots\ldots (3),$$

deduced from the equation (1), and which are sufficient to

determine the expansion of Fx, considered as a function of u and h in powers of h.

Consider generally the symbolical expression

$$\phi\left(h\,\frac{h}{dh}\right)\Xi(h) \ldots\ldots\ldots (4),$$

$\phi\left(h\,\frac{d}{dh}\right)$ involving in general symbols of operation relative to any of the other variables entering into $\Xi(h)$. Then, if $\Xi(h)$ be expansible in the form

$$\Xi(h) = \Sigma(Ah^m) \ldots\ldots\ldots (5),$$

it is obvious that

$$\phi\cdot\left(h\,\frac{d}{dh}\right)\Xi(h) = \Sigma\{\phi m\,.\,(Ah^m)\} = \Sigma\{(\phi m\,.\,A)\,h^m\}\,..(6).$$

For instance, (u) representing a variable contained in the function $\Xi(h)$, and taking a particular form of $\phi\left(h\,\frac{d}{dh}\right)$,

$$\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\cdot\,\Xi(h) = \Sigma\left(\frac{d^m\,.\,A}{du^m}\,.\,h^m\right)\ldots\ldots\ldots (7).$$

From which it is easy to demonstrate

$$\frac{d}{du}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\cdot\,\Xi(h)\right\} = \frac{1}{h}\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\cdot\,\{h\,\Xi(h)\}\ldots.(8),$$

$$\frac{d}{dh}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\cdot\,\Xi(h)\right\} = \frac{1}{h}\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\cdot\,\{h\,\Xi'(h)\}\ldots.(9),$$

where $\Xi' h$ denotes $\frac{d}{dh}\,\Xi(h)$, as usual. Hence also

$$\frac{d}{du}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\cdot\,\Xi' h\right\} = \frac{d}{dh}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}}\cdot\,\Xi h\right\}\ldots.(10),$$

of which a particular case is

$$\frac{d}{du}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}\cdot F'u\,fu\,e^{hfu}\right\} = \frac{d}{dh}\left\{\left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}\cdot F'u\,.\,e^{hfu}\right\}..(11).$$

Also, $$\left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}(F'u\,e^{hfu}) = Fu \quad \text{for} \quad h = 0\,..(12).$$

Hence the form in question for Fx is

$$Fx = \left(\frac{d}{du}\right)^{h\frac{d}{dh}-1}(F'u\,e^{hfu}) \ldots\ldots (13);$$

from which, differentiating with respect to u, and writing F instead of F',

$$\frac{Fx}{1-hf'x} = \left(\frac{d}{du}\right)^h \frac{d}{dh}\,(Fu\,e^{hfu}) \ldots\ldots (14),$$

a well-known form of Lagrange's theorem, almost equally important with the more usual one. It is easy to deduce (13) from (14). To do this, we have only to form the equation

$$\frac{-hFx\,.\,f'x}{1-hf'x} = -h\left(\frac{d}{du}\right)^h \frac{d}{dh}\,(Fuf'u\,e^{hfu}) \ldots (15),$$

deduced from (14) by writing $Fxf'x$ for fx, and adding this to (14),

$$\begin{aligned} Fx &= \left(\frac{d}{du}\right)^h \frac{d}{dh}\,(Fu\,e^{hfu}) - h\left(\frac{d}{du}\right)^h \frac{d}{dh}\,(Fuf'u\,e^{hfu}) \\ &= \left(\frac{d}{du}\right)^h \frac{d}{dh}{}^{-1}\left\{\frac{d}{du}(Fu\,.\,e^{hfu}) - h\,.\,f'u\,Fu\,.\,e^{hfu}\right\} \\ &= \left(\frac{d}{du}\right)^h \frac{d}{dh}{}^{-1}\,(F'u\,e^{hfu}) \ldots\ldots\ldots\ldots\ldots (16). \end{aligned}$$

In the case of several variables, if

$$\left.\begin{aligned} x &= u + h\,f\,(x,\ x_1\ldots) \\ x_1 &= u_1 + h_1 f_1 (x,\ x_1\ldots) \end{aligned}\right\} \ldots\ldots (17),$$

writing for shortness

$$F,\ f,\ f_1\ldots \text{ for } F(u,\ u_1\ldots),\ f(u,\ u_1\ldots),\ f_1(u,\ u_1\ldots),$$

$$\frac{F(x,\ x_1\ldots)}{\{1-hf'(x)\}\{1-h_1 f_1'(x_1)\ldots\}} = \left(\frac{d}{du}\right)^h \frac{d}{dh}\left(\frac{d}{du_1}\right)^{h_1} \frac{d}{dh_1}\ldots(Fe^{hf+h_1 f_1+\ldots})\ldots (18),$$

$$\{\text{where } f'(x) = \frac{d}{dx}\,.f(x,\ x_1\ldots),\ \&c.\}$$

or the coefficient of $h^n\,.\,h_1^{n_1}\ldots$ in the expansion of

$$\frac{F(x,\ x_1\ldots)}{\{1-hf'(x)\}\ \{1-h_1 f_1'(x_1)\}\ldots} \ldots\ldots\ldots (19)$$

is $$\frac{1}{1.2\ldots n\,.\,1.2\ldots n_1}\left(\frac{d}{du}\right)^n\left(\frac{d}{du_1}\right)^{n_1}\ldots Ff^n\,.\,f_1^{n_1}\ldots\ldots (20).$$

From the formula (18), a formula may be deduced for the expansion of $F(x,\ x_1\ldots)$, in the same way as (13) was deduced from (14), but the result is not expressible in a simple form by this method. An apparently simple form has indeed been given for this expansion by Laplace, *Mécanique Celeste*, tom. I. p. 176; but the expression there given for the general

term, requires first that certain differentiations should be performed, and then that certain changes should be made in the result, quantities $z, z' \ldots$, which are to be changed into $z'', z_1'' \ldots$; in other words, the general term is not really expressed by known symbols of operation only. The formula (18) is probably known, but I have not met with it anywhere.

X.—NOTE ON ORTHOGONAL ISOTHERMAL SURFACES.

THE object of this article is to state an important question relative to Orthogonal Isothermal Surfaces, which does not seem to have been yet considered generally.

It is known that if a system of surfaces be given, there are two, and only two, other systems cutting one another and the first system at right angles. Lamé has proved many general properties of such conjugate systems of surfaces, and has also considered particularly the case in which each of the three systems is a series of isothermal surfaces. He has nowhere however proved the possibility of the existence of three such systems of surfaces in general, or, in other words, he has not shown that if one system be isothermal, the two others which are determined by it are isothermal also. There is however a considerable probability that this proposition is generally true. For, in the first place, it is true, as will be shown below, of any system whatever of isothermal *cylindrical* surfaces, and it is also true of all isothermal surfaces of the second order, whether cylindrical or not. As these two cases, the only ones which have as yet been discussed, are very distinct in their nature, they afford a considerable presumptive evidence of the truth of the general proposition.

The case of isothermal surfaces of the second order has been fully discussed by Lamé, in a paper on Isothermal Surfaces in Liouville's Journal, (vol. II. p. 147), where he has shown that all *confocal* surfaces of the second order are isothermal. Now the two systems of orthogonal surfaces conjugate to a given series of confocal surfaces of the second order are also confocal surfaces of the second order, and therefore they are likewise isothermal. Hence the proposition is true for isothermal surfaces of the second order.

It may be proved in the following manner to be true for a series of isothermal cylindrical surfaces.

Of the two systems of orthogonal surfaces, conjugate to the given series of cylindrical surfaces, one is a series of cylindrical surfaces cutting them at right angles, and having their generating lines parallel to those of the first system, and the

other is a series of planes perpendicular to the cylinders. Now a series of parallel planes is obviously a system of isothermal surfaces, and it therefore only remains to be proved that, if one series of cylindrical surfaces be isothermal, the series which cuts them at right angles will be isothermal also.

To prove this, let us suppose any two surfaces of the first system to be retained at uniform temperatures, and let v be the temperature of any point (xy) between them. The equation of any one of the isothermal surfaces will be

$$v = a \ldots\ldots\ldots\ldots\ldots (1),$$

which is therefore the general equation, comprehending all surfaces of the first series.

By the equation of equilibrium of heat, moving in two directions, we must have

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

Now let

$$v_1 = a_1 \ldots\ldots\ldots\ldots\ldots (3)$$

be the general equation of the system of cylindrical surfaces, which cuts the first system at right angles.

Hence

$$\frac{dv}{dx}\frac{dv_1}{dx} + \frac{dv}{dy}\frac{dv_1}{dy} = 0 \ldots\ldots\ldots\ldots (4),$$

Let

$$\frac{dv_1}{dx} = k\frac{dv}{dy} \ldots\ldots\ldots (a),$$

therefore

$$\frac{dv_1}{dy} = -k\frac{dv}{dx} \ldots\ldots\ldots (b).$$

In these equations k is arbitrary, with the exception that it must be so chosen that $\frac{dv_1}{dx}\,dx + \frac{dv_1}{dy}\,dy$ shall be a complete differential. Hence we must have

$$\frac{d}{dy}(a) - \frac{d}{dx}(b) = 0,$$

and therefore

$$k\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}\right) + \frac{dk}{dx}\frac{dv}{dx} + \frac{dk}{dy}\frac{dv}{dy} = 0,$$

or, on account of (2),

$$\frac{dk}{dx}\frac{dv}{dx} + \frac{dk}{dy}\frac{dv}{dy} = 0.$$

This equation is satisfied if $k = 1$, and therefore we may use this value for k in (a) and (b). Hence

$$\frac{dv_1}{dx} = \frac{dv}{dy} \quad \ldots\ldots\ldots\ldots\ldots (c),$$

$$\frac{dv_1}{dy} = -\frac{dv}{dx} \quad \ldots\ldots\ldots\ldots\ldots (d).$$

These equations determine a form of v_1 which satisfies equation (4), the only condition to which v_1 is subject, in order that $v_1 = a_1$ may be the equation to the series of cylinders cutting the first series at right angles.

Now $\frac{d}{dx}(c) + \frac{d}{dy}(d)$ gives

$$\frac{d^2v_1}{dx^2} + \frac{d^2v_1}{dy^2} = 0 \quad \ldots\ldots\ldots\ldots (5).$$

Hence a function v_1 of x and y, which satisfies (5), may be found such that $v_1 = a_1$ represents the series of cylindrical surfaces cutting the first series at right angles. Hence the second system is isothermal.

The theorem which has just been proved, was first given by Lamé in a digression on Orthogonal Surfaces, contained in a paper entitled "*Mémoire sur les Lois de l'Equilibre du Fluide Ethéré,*" in the *Journal de l'Ecole Polytechnique*, (vol. III. cahier XXIII). His proof however is deduced from some general properties of orthogonal surfaces which he has been discussing, and in a subsequent paper on the motion of heat in cylindrical bodies, (Liouville's Journal, vol. I.) he has not noticed the proposition at all.

P. Q. R.

XI.—ON FOURIER'S THEOREM.

The following demonstration of Fourier's Theorem for the transformation of a function may be interesting, as connecting it more directly with the Calculus of Operations and the ordinary series for the development of functions. It is not offered as eluding or overcoming the difficulties which seem to attend every demonstration of this important theorem, but it at least shows distinctly the nature of the difficulty to be overcome.

Let us consider the integral

$$\int da\, f(a) \cos q\,(x - a) = \int da\, f(a)\, \varepsilon^{-a\frac{d}{dx}} \cos qx,$$

by Taylor's theorem. But by the property of a differential of the product of two functions

$$\int da f(a)\, \varepsilon^{-\alpha \frac{d}{da}} = \left(\frac{d}{da} + \frac{d'}{da}\right)^{-1} f(a)\, \varepsilon^{-\alpha \frac{d}{da}},$$

where the accented letter refers to $f(a)$, and the unaccented to $\varepsilon^{-\alpha \frac{d}{da}}$. Expanding the second side by the theorem of Bernoulli, we have

$$\int da f(a) \varepsilon^{-\alpha \frac{d}{da}} = \left\{\left(\frac{d}{da}\right)^{-1} - \left(\frac{d}{da}\right)^{-2} \frac{d'}{da} + \left(\frac{d}{da}\right)^{-3}\left(\frac{d'}{da}\right)^{2} - \&c.\right\} f(a)\, \varepsilon^{-\alpha \frac{d}{da}}.$$

And since

$$\left(\frac{d}{da}\right)^{-1} \varepsilon^{-\alpha \frac{d}{da}} = \int da\, \varepsilon^{-\alpha \frac{d}{da}} = -\left(\frac{d}{dx}\right)^{-1} \varepsilon^{-\alpha \frac{d}{da}}, \text{ we have}$$

$$\int da f(a) \varepsilon^{-\alpha \frac{d}{da}} = -\left(\frac{d}{dx}\right)^{-1}\left\{1 + \left(\frac{d}{dx}\right)^{-1} \frac{d}{da} + \left(\frac{d}{dx}\right)^{-2}\left(\frac{d}{da}\right)^{2} + \&c.\right\} f(a) \varepsilon^{-\alpha \frac{d}{dx}}.$$

Applying the operations on both sides to $\cos qx$, we find

$$\int da f(a) \cos q(x-a) = -\left(\frac{d}{dx}\right)^{-1}\left\{1 + \left(\frac{d}{dx}\right)^{-1} \frac{d}{da} + \&c.\right\} f(a) \cos q(x-a).$$

On taking this between the limits of $a = 0$ and $a = \infty$, and assuming that $\cos \infty$ is a zero of an order sufficient to destroy $f(\infty)$, this gives us

$$\int_0^\infty da f(a) \cos q(x-a) = \left(\frac{d}{dx}\right)^{-1}\left\{f(0) + \left(\frac{d}{dx}\right)^{-1} f'(0) + \&c.\right\} \cos qx,$$

where $f^{(n)}(0) = \left(\frac{d}{da}\right)^n f(a)$ when $a = 0$.

Now if $u = \int_0^\infty dq \int_0^\infty da f(a) \cos q(x-a)$, we have

$$u = \left(\frac{d}{dx}\right)^{-1}\left\{f(0) + \left(\frac{d}{dx}\right)^{-1} f'(0) + \&c.\right\} \int_0^\infty dq \cos qx.$$

But $\left(\frac{d}{dx}\right)^{-1} \int_0^\infty dq \cos qx = \int_0^\infty \frac{dq}{q} \sin qx = \frac{\pi}{2}$,

and therefore

$$\left(\frac{d}{dx}\right)^{-(r+1)} \int_0^\infty dq \cos qx = \frac{\frac{1}{2}\pi x^r}{1.2 \ldots r},$$

so that $u = \frac{\pi}{2}\left\{f(0) + \frac{x}{1} f'(0) + \frac{x^2}{1.2} f''(0) + \&c.\right\}$

The part within brackets is, by Maclaurin's theorem, equal to $f(x)$, and so

$$\int_0^\infty dq \int_0^\infty da\, f(a) \cos q\,(x - a) = \frac{\pi}{2} f(x),$$

which is Fourier's theorem.

It is to be observed, that the proof depends essentially on the assumption that the expression

$$f(a) \cos q\,(x - a)$$

vanishes when $a = \infty$, which implies that $\cos \infty$ is a zero of a higher order than the infinity which may be the value of $f(\infty)$. This agrees with a remark made by Professor De Morgan, that Defler's verification depends on the assumption that $\sin \infty$ is a zero of the same order as $\epsilon^{-\infty}$. (See his *Differential Calculus*, p. 628, note.)

G.

XII.—MATHEMATICAL NOTES.

1. *Stability of Eccentricities and Inclinations.* The equation for proving the stability of the eccentricites and inclinations of the planetary orbits may, as has been shown by Laplace, be deduced from the principle of the conservation of areas, joined to the fact of the invariability of the major axes.

Let m be the mass of a planet, h be twice the area described by the radius vector about the sun in an unit of time projected on the ecliptic, and i the inclination of its orbit to the ecliptic; then, by the principle of the conservation of areas,

$$\Sigma (mh) = \text{const.}$$

Every term under the sign of Summation is positive, because all the planets move round the sun in the same direction. But

$$h = \{a(1 - e^2)\}^{\frac{1}{2}} \cos i = a^{\frac{1}{2}}(1 - e^2)^{\frac{1}{2}}(1 + \tan^2 i)^{-\frac{1}{4}}.$$

Now e and i are small at the present time; hence, if we neglect their fourth power and the products of their squares, we have

$$h = a^{\frac{1}{2}} \{1 - \tfrac{1}{2}e^2 - \tfrac{1}{2} \tan^2 i\}.$$

Hence $$\Sigma \{ma^{\frac{1}{2}}(1 - \tfrac{1}{2}e^2 - \tfrac{1}{2} \tan^2 i)\} = \text{const.}$$

But since the major axes have no secular inequalities

$$\Sigma (ma^{\frac{1}{2}}) = \text{const.};$$

hence the preceding equation is equivalent to

$$\Sigma (ma^{\frac{1}{2}} e^2 + ma^{\frac{1}{2}} \tan^2 i) = \text{const.}$$

Now the left-hand side of the equation being small at the present time, the second side is also small, and therefore the first side is always small, and therefore

$$\Sigma\,(ma^{\frac{1}{2}}\,e^2) \text{ and } \Sigma\,(ma^{\frac{1}{2}}\tan^2 i)$$

are both always small. ε.

2. *Mnemonic Rule.*—The following mnemonic rule for the "Cotangent" formula in spherical trigonometry may be found useful.

If in any spherical triangle four parts be taken in succession, as for example, A, c, B, a, consisting of two means c, B, and two extremes A, a; then "The product of the cosines of the two means is equal to the sine of the mean *side* × cotangent of the extreme *side* – sine of the mean *angle* × cotangent of extreme *angle*." That is,

$$\cos c \cos B = \sin c \cot a - \sin B \cot A.$$

The negative sign may be got rid of by taking as the parts the two angles and the *supplements* of the two sides.

π.

3. To shew that the greatest and least radii of any plane section of the surface of elasticity are at right angles.

We have (*Math. Jour.* vol. I. p. 8.)

$$x = \frac{Al}{r^2 - a^2}, \quad y = \frac{Am}{r^2 - b^2}, \quad z = \frac{An}{r^2 - c^2},$$

$$\text{and} \quad \frac{l^2}{r^2 - a^2} + \frac{m^2}{r^2 - b^2} + \frac{n^2}{r^2 - c^2} = 0.$$

Let r_1 and r_2 be the two roots of this equation;

then

$$\frac{l^2}{r_1^2 - a^2} + \frac{m^2}{r_1^2 - b^2} + \frac{n^2}{r_1^2 - c^2} = 0,$$

$$\frac{l^2}{r_2^2 - a^2} + \frac{m^2}{r_2^2 - b^2} + \frac{n^2}{r_2^2 - c^2} = 0.$$

Subtracting,

$$\frac{l^2\,(r_1^2 - r_2^2)}{(r_1^2 - a^2)\,(r_2^2 - a^2)} + \frac{m^2\,(r_1^2 - r_2^2)}{(r_1^2 - b^2)\,(r_2^2 - b^2)} + \frac{n^2\,(r_1^2 - r_2^2)}{(r_1^2 - c^2)\,(r_2^2 - c^2)} = 0;$$

dividing by $r_1^2 - r_2^2$, and multiplying by A^2,

$$x_1x_2 + y_1y_2 + z_1z_2 = 0;$$

which, as $x_1y_1z_1$, $x_2y_2z_2$, are proportional to the direction cosines of the lines, shows that they are at right angles to each other.

4. *Problem in the Papers of* 1842. A quadrilateral, composed of four unequal beams jointed together at the extremities, is compressed by a given force in the direction of one diagonal: find the force in the direction of the other diagonal which will resist the compression.

Let the quadrilateral be $ABCD$, and let the diagonals AC, BD, intersect in O. Then if P be the compressing force in AC, and Q the resisting force in BD, and X be the force in AD, we have, considering AB as a lever turning round B as a fulcrum, and acted on by forces P and X at A,

$$P \sin BAC = X \sin BAD.$$

In the same way, considering CD as a lever turning on a fulcrum at C, and acted on by Q and X at D, we have

$$Q \sin BDC = X \sin CDA;$$

$$\text{therefore} \quad \frac{P}{Q} = \frac{\sin BDC}{\sin BAC} \cdot \frac{\sin BAD}{\sin CDA}.$$

But in the triangle ABD we have

$$\frac{\sin BAD}{\sin ABD} = \frac{BD}{AD};$$

in the triangle ACD, $\dfrac{\sin CDA}{\sin ACD} = \dfrac{AC}{AD}$;

$$\text{therefore} \quad \frac{P}{Q} = \frac{BD}{AC} \cdot \frac{\sin ABD}{\sin ACD} \cdot \frac{\sin BDC}{\sin BAC}.$$

In the triangle ABO, $\dfrac{\sin ABD}{\sin BAC} = \dfrac{AO}{BO}$;

in the triangle CDO, $\dfrac{\sin BDC}{\sin ACD} = \dfrac{CO}{DO}$;

$$\text{hence} \quad \frac{P}{Q} = \frac{BD}{AC} \cdot \frac{AO \cdot CO}{BO \cdot OD}.$$

If the quadrilateral be a parallelogram, $AO = OC$ and $BO = OD$, so that

$$\frac{P}{Q} = \frac{AC}{BD},$$

or the forces are directly as the diagonals.

γ.

END OF VOL. III.

METCALFE AND PALMER, PRINTERS, CAMBRIDGE.